

ISSN: 2281-1346



Department of Economics and Management

**DEM Working Paper Series**

**Equilibrium and Optimality in Gale-von  
Neumann Models**

Giorgio Giorgi  
(Università di Pavia)

Cesare Zuccotti  
(Università di Pavia)

**# 118 (04-16)**

Via San Felice, 5  
I-27100 Pavia  
<http://epmq.unipv.eu/site/home.html>

**April 2016**

# Equilibrium and Optimality in Gale-von Neumann Models

Giorgio Giorgi<sup>1</sup> and Cesare Zuccotti<sup>2</sup>

## Summary

We give an overview of the main properties concerning equilibrium, optimality and turnpike theorems for a Gale-von Neumann economic growth model.

## Key words

Gale - von Neumann models; intertemporal efficiency; turnpike theorems.

## 1. Introduction

The aim of the present paper is to give an overview on the main equilibrium and optimality properties of a Gale - von Neumann growth model. The classical von Neumann model (von Neumann (1945-46)) of an expanding economy has been generalized by Gale (1956), who considered production as expressed by a closed convex cone, instead of by a pair of nonnegative matrices. The Gale - von Neumann model has been subsequently considered by Karlin (1959) and by many other authors, such as, e. g., Drandakis (1966), Furuya and Inada (1962), Nikaido (1964,1968), Morishima (1964), Radner (1961), Winter (1965, 1967), Makarov and Rubinov (1970, 1977), Evstigneev and Shank-Hoppé (2006), Giorgi (2004), Wan (1971).

We base our treatment mainly on Cruceanu (1978), Makarov and Rubinov (1977), Nikaido (1968), Morishima (1964), Takayama (1985).

<sup>1</sup> Department of Economics and Management, Via S. Felice, 5 - 27100 Pavia, (Italy). E-mail: ggiorgi@eco.unipv.it

<sup>2</sup> Department of Economics and Management, Via S. Felice, 5 - 27100 Pavia, (Italy). E-mail: czuccotti@eco.unipv.it

The paper is organized as follows.

Section 2 is concerned with the description of the Gale-von Neumann model.

Section 3 treats the existence of equilibrium solutions for the said model.

Section 4 is concerned with growth paths (or growth trajectories), balanced growth paths and maximal balanced growth paths.

Section 5 is concerned with the optimal final paths (or trajectories).

Section 6 is concerned with efficient growth.

Section 7 treats the main Turnpike Theorems (for a Gale-von Neumann model).

We adopt the following conventions for vector inequalities. Let  $a, b \in \mathbb{R}^n$ ; then  $a \geq b$  means  $a_i \geq b_i, \forall i = 1, \dots, n$ ;  $a \geq b$  means  $a \geq b$  and  $a \neq b$ ;  $a > b$  means  $a_i > b_i, \forall i = 1, \dots, n$ . If  $b = [0]$  (zero vector of  $\mathbb{R}^n$ ), then the vector  $a$  is, respectively, a nonnegative, a semipositive, a positive vector. The same convention is adopted for the comparison of two matrices of the same order. The norm  $\|x\|$  of a vector  $x$  in  $\mathbb{R}^n$  is  $(x^\top x)^{1/2}$ . If  $A$  is a matrix of order  $(m, n)$ , by  $A_i, i = 1, \dots, m$ , we denote its  $i$ -th row and by  $A^j, j = 1, \dots, n$ , we denote its  $j$ -th column. The zero matrix is denoted by  $[0]$ .

## 2. Assumptions on Gale-von Neumann Growth Models

The model to be considered involves  $n$  commodities; at the beginning of each period some amounts  $x_1, \dots, x_n$  of the various commodities are used as inputs, in order to obtain, at the end of the period, the outputs described by the commodities  $y_1, \dots, y_n$ . The set of technologically possible pairs  $(x, y)$  is denoted by  $T$ ; obviously  $T \subset \mathbb{R}_+^n \times \mathbb{R}_+^n$  and the pair  $(x, y)$  is technologically possible if and only if  $(x, y) \in T$ . On the transformation set (or technology set)  $T$  usually the following assumptions are made.

**Assumption (T<sub>1</sub>).** The set  $T$  is a closed convex cone in  $\mathbb{R}_+^n \times \mathbb{R}_+^n$ . This assumption translates the property of *proportionality* of the transformation, i. e. if  $(x, y) \in T$ , then  $(\lambda x, \lambda y) \in T$  for any  $\lambda \geq 0$  (“constant returns to scale”); of *additivity* of the transformation, i. e. if  $(x, y) \in T$  and  $(x', y') \in T$ , then  $(x + x', y + y') \in T$ ; and of *closedness* of the transformation, i. e. also the boundary points of  $T$  are feasible activities.

**Assumption (T<sub>2</sub>).** If  $([0], y) \in T$ , then  $y = [0]$ . In other words, it is impossible to produce something from nothing (impossibility of the “Land of Cockaigne”).

**Assumption (T<sub>3</sub>).** For any  $x \geq [0]$  there exists a vector  $y \geq [0]$  such that  $(x, y) \in T$ . In other words, this assumption makes possible a production with any initial endowment of available stocks.

**Assumption (T<sub>4</sub>).** For any  $i = 1, \dots, n$ , there exists  $(x^i, y^i) \in T$  such that  $y_i^i > 0$ . That is, every commodity can be produced. In view of (T<sub>1</sub>), the present assumption is equivalent to: there exists a process  $(\hat{x}, \hat{y}) \in T$  such that  $\hat{y} > [0]$ .

**Assumption (T<sub>5</sub>).** If  $(x, y) \in T$ , then  $x^* \geq x$  and  $[0] \leq y^* \leq y$  imply that  $(x^*, y^*) \in T$ . This assumption implies that disposal activity is costless: a smaller output is certainly possible with a larger input.

**Remark 1.** Assumptions (T<sub>1</sub>) and (T<sub>5</sub>) imply assumption (T<sub>3</sub>). Indeed, from (T<sub>1</sub>) we get  $([0], [0]) \in T$ ; on the other hand, from (T<sub>5</sub>) we get  $(x, [0]) \in T$ ,  $x \geq [0]$ .

**Remark 2.** The polyhedral technology set of the classical von Neumann model

$$T = \{(x, y) : x = Av, y = Bv, v \geq [0]\}$$

is obviously a special case of the technology set  $T$  introduced above. Here  $A \geq [0]$  and  $B \geq [0]$  are, respectively, the inputs matrix and the outputs matrix, of order  $(n, m)$ . Assumption (T<sub>1</sub>) is obviously satisfied, as this cone is polyhedral. If we assume, as in Kemeny, Morgenstern and Thompson (1956), the condition

$$(KMT 1) \quad A^j \geq [0], j = 1, \dots, m,$$

then assumption (T<sub>2</sub>) is satisfied. If, following the same authors, we assume the condition

$$(KMT 2) \quad B_i \geq [0], i = 1, \dots, n,$$

then assumption (T<sub>4</sub>) is satisfied. We can also extend the definition of the technology set, in order that also assumption (T<sub>5</sub>) is satisfied. For this purpose we define the following technology set

$$T' = \{(x, y) : x \geq Av, [0] \leq y \leq Bv, v \geq [0]\}.$$

We may call  $T'$  *von Neumann normalized technology set*. Obviously,  $T \subset T'$ , assumption (T<sub>5</sub>) is verified by  $T'$  and again (KMT 1) and (KMT 2) imply, respectively, (T<sub>2</sub>) and (T<sub>4</sub>).

It is easy to see that also the Leontief-von Neumann models introduced by Gale (1960), Karlin (1959), Los (1971), are special cases of the technology set  $T$ . It is the same for the Leontief-Morishima-von Neumann model introduced by Morishima (1961, 1964).

Let  $T \subset \mathbb{R}_+^n \times \mathbb{R}_+^n$  a technology set satisfying assumptions (T<sub>1</sub>) and (T<sub>2</sub>) and let us suppose that  $T$  is not a trivial technology, that is that it does not contain only the pair  $([0], [0])$ , which would be equivalent to the “inactivity” of all processes. We define the *rate of expansion* of the process  $(x, y) \in T$  as the largest value of  $\alpha$  such that  $y \geq \alpha x$ :

$$\alpha(x, y) = \max_{\alpha \in \mathbb{R}} \{\alpha : y \geq \alpha x\},$$

where  $(x, y) \neq [0]$ . By definition we put  $\alpha([0], [0]) = +\infty$ . Several authors call  $\alpha(x, y)$  *factor of expansion* or *factor of growth* and call rate of expansion the number  $\alpha - 1$ . It is evident that

$\alpha(x, y) \geq 0$ . We can also define the real-valued function  $\alpha(x, y)$  as follows. Let  $(x, y) \neq ([0], [0])$  and define

$$I_{(x,y)} = \{i : x_i + y_i > 0\}.$$

For any  $i \in I_{(x,y)}$  we define as follows the rate of expansion of the  $i$ -th commodity in process  $(x, y)$  and denoted  $\alpha_i(x, y)$  :

$$\alpha_i(x, y) = \begin{cases} \frac{y_i}{x_i}, & \text{if } x_i > 0 \\ +\infty, & \text{if } x_i = 0. \end{cases} \quad (1)$$

Then

$$\alpha(x, y) = \min_{i \in I_{(x,y)}} \alpha_i(x, y) \quad (2)$$

or

$$\alpha(x, y) = \max \{\alpha : \alpha x \leq y\}. \quad (3)$$

Since  $\alpha(x, y)$  is a function of  $(x, y)$ , the expansion rate varies from process to process; contrary to what asserted, e. g., by Morishima (1964) and by Nicola (1976) and to what (perhaps) implicitly supposed by Gale (1960), the function  $\alpha : T \setminus \{[0], [0]\} \rightarrow \mathbb{R}_+$  is *not* continuous but only upper semicontinuous. See also Glycopantis (1970) and Giorgi (2004).

**Lemma 1.** The function  $\alpha(x, y)$  is positively homogeneous of degree zero and upper semicontinuous.

**Proof.** Let be given  $(x, y) \in T \setminus \{[0], [0]\}$  and  $\lambda > 0$ . For each  $i \in I_{(x,y)}$  we have

$$\alpha_i(\lambda x, \lambda y) = \alpha_i(x, y). \quad (4)$$

As in processes  $(\lambda x, \lambda y)$  and  $(x, y)$  the same commodities are implied, from (4) we get

$$\alpha(\lambda x, \lambda y) = \min_{i \in I_{(\lambda x, \lambda y)}} \alpha_i(\lambda x, \lambda y) = \min_{i \in I_{(x,y)}} \alpha_i(x, y) = \alpha(x, y),$$

that is the function  $\alpha(x, y)$  is positively homogeneous of degree zero. Let us now consider a sequence  $\{(x^n, y^n)\}_{n=1}^{\infty} \subset T \setminus \{[0], [0]\}$  such that  $(x^n, y^n) \rightarrow (x, y)$  for  $n \rightarrow \infty$ . Let  $\bar{\alpha}$  be a limit point of the numerical sequence  $\{\alpha(x^n, y^n)\}_{n=1}^{\infty}$ . Then, there exists a subsequence  $\{(x^{n_k}, y^{n_k})\}_{k=1}^{\infty} \subset \{(x^n, y^n)\}_{n=1}^{\infty}$  such that  $\alpha(x^{n_k}, y^{n_k}) \rightarrow \bar{\alpha}$  when  $k \rightarrow \infty$ . From the inequality

$$\alpha(x^{n_k}, y^{n_k}) x^{n_k} \leq y^{n_k},$$

when  $k \rightarrow \infty$  we find the limit

$$\bar{\alpha} x \leq y.$$

From this result and from (3) we have  $\bar{\alpha} \leq \alpha(x, y)$ , but as  $\bar{\alpha}$  is an arbitrary limit point of the sequence  $\{\alpha(x^n, y^n)\}_{n=1}^{\infty}$ , we deduce

$$\limsup_{n \rightarrow \infty} \alpha(x^n, y^n) \leq \alpha(x, y).$$

Therefore the function  $\alpha(x, y)$  is upper semicontinuous on  $T \setminus \{[0], [0]\}$ . □

It is well-known that an upper semicontinuous function admits a global maximum over any compact set. As the set  $T \cap \{(x, y) : (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n, \|(x, y)\| = 1\}$  is compact,  $\alpha(x, y)$  admits a global maximum on this set. This maximum is also the maximum on the set  $T \setminus \{[0], [0]\}$ , as  $\alpha$  is a positively homogeneous function of degree zero on this set. We denote

$$\alpha(T) = \max_{\substack{(x,y) \in T, \\ \|(x,y)\|=1}} \alpha(x, y) = \max_{\substack{(x,y) \in T, \\ (x,y) \neq ([0],[0])}} \alpha(x, y). \quad (5)$$

**Definition 1.** The number defined by (5) is said to be the *rate of expansion of the technology*  $T$  or *von Neumann expansion rate of  $T$* . Any process  $(\bar{x}, \bar{y}) \in T$  such that  $\bar{y} \geq \alpha(T)\bar{x}$  is called *optimal process* or *von Neumann process*.

**Remark 3.** Let  $Q = \{(x, y) : (x, y) \in T, \alpha(x, y) = \alpha(T)\}$  the set of all von Neumann processes for a technological set  $T$ . Therefore a process  $(x, y)$  belongs to  $Q$  if and only if  $(x, y) \neq ([0], [0])$ ,  $\alpha(T)x \leq y$ ; it is easy to verify that the elements of the set  $Q$  verify the following properties:

- a)  $(x', y') \in Q, (x'', y'') \in Q$  implies  $(x' + x'', y' + y'') \in Q$ ;
- b) If  $(x, y) \in Q$ , then  $(\lambda x, \lambda y) \in Q$  for any number  $\lambda > 0$ .

**Remark 4.** The growth rate  $\alpha(T)$  has been defined for a technology  $T$  which satisfies assumptions  $(T_1)$  and  $(T_2)$ . These assumptions, however, do not assure the positivity of  $\alpha(T)$ . If, besides  $(T_1)$  and  $(T_2)$ , also assumption  $(T_4)$  is imposed, then  $\alpha(T) > 0$ . Indeed, let  $(\hat{x}, \hat{y}) \in T$  such that  $\hat{y} > [0]$ . From (1) and (2) we get  $\alpha(\hat{x}, \hat{y}) > 0$  and therefore it holds  $\alpha(T) > 0$ .

**Lemma 2.** Let  $(T_1)$  and  $(T_2)$  be verified. There exists a vector  $\bar{p} \geq [0]$  such that

$$\bar{p}y - \alpha(T)\bar{p}x \leq 0, \quad \forall (x, y) \in T. \quad (6)$$

**Proof.** Let us consider the following set of  $\mathbb{R}^n$  :

$$L = \{y - \alpha(T)x : (x, y) \in T\}.$$

Being  $T$  a convex cone, the set  $L$  is a convex cone too. From the definition of  $\alpha(T)$  we obtain

$$L \cap \text{int } \mathbb{R}_+^n = \emptyset.$$

It results therefore that the sets  $L$  and  $\mathbb{R}_+^n$  can be separated by a hyperplane passing through the origin. Therefore, there exists a vector  $\bar{p} \in \mathbb{R}^n$ ,  $\bar{p} \neq [0]$ , such that

$$\begin{aligned}\bar{p}z &\geq 0, \quad \forall z \in \mathbb{R}_+^n. \\ \bar{p}z &\leq 0, \quad \forall z \in L.\end{aligned}$$

From the first inequality we obtain  $\bar{p} \geq [0]$  and from the second inequality relation (6) follows.  $\square$

The vector  $\bar{p}$  of relation (6) is also called *von Neumann price vector*; whenever  $\bar{p}x > 0$ , the ratio  $(\bar{p}y)/(\bar{p}x)$  may be considered as the “profit ratio” of process  $(x, y)$ . Lemma 2 states that this does not exceed the maximum expansion rate  $\alpha(T)$ .

### 3. Equilibrium Solutions for the Gale-von Neumann Model

Following Los (1971), who mainly analyzed the classical (polyhedral) von Neumann model, we give the following definition.

**Definition 2.** A quadruplet  $((\bar{x}, \bar{y}), \bar{p}, \alpha)$ , where  $(\bar{x}, \bar{y}) \in T$ ,  $\bar{x}, \bar{p} \in \mathbb{R}_+^n$ ,  $\alpha \in \mathbb{R}_+$ , is called an *equilibrium solution* for the technology  $T$  if it satisfies the following relations:

$$\alpha\bar{x} \leq \bar{y}, \tag{7}$$

$$\bar{p}y \leq \alpha\bar{p}x, \quad \forall (x, y) \in T, \tag{8}$$

$$\bar{p}\bar{y} > 0. \tag{9}$$

From relations (7), (8) and (9) it results at once that for each equilibrium solution  $((\bar{x}, \bar{y}), \bar{p}, \alpha)$  we have  $\alpha > 0$ ,  $\bar{x} \geq [0]$ ,  $\bar{y} \geq [0]$ ,  $\alpha\bar{p}\bar{x} = \bar{p}\bar{y}$ . It is well-known (see, e. g., Kemeny, Morgenstern and Thompson (1956)) that the classical von Neumann model, where matrices  $A$  and  $B$  satisfy properties (KMT 1) and (KMT 2), admits at least the equilibrium solution where  $\alpha = \alpha(T)$ . Besides the quoted paper of Kemeny, Morgenstern and Thompson, see also, e. g., Giorgi and Meriggi (1987, 1988), Gale (1972), Howe (1960), Los (1971).

**Definition 3.** An equilibrium solution of the form  $((\bar{x}, \bar{y}), \bar{p}, \alpha(T))$  is called *von Neumann equilibrium solution*.

It can be proved that the assumptions (KMT 1) and (KMT 2) on matrices  $A$  and  $B$  for the classical von Neumann model imply assumptions (T<sub>1</sub>), (T<sub>2</sub>), and (T<sub>4</sub>); see, e. g., Takayama (1985).

Now our problem is: are assumptions (T<sub>1</sub>), (T<sub>2</sub>) and (T<sub>4</sub>) sufficient for the existence of a von Neumann equilibrium solution for the general Gale-von Neumann model? The following example, due to Hülsmann and Steinmetz (1972), shows that, for a non-polyhedral technology,

the above assumptions in general do not assure the existence of a von Neumann equilibrium solution.

**Example 1.** Let  $T \subset \mathbb{R}_+^3 \times \mathbb{R}_+^3$  the convex cone generated by the set

$$T_0 = \left\{ (1, 1, t; 3, 2 \pm \sqrt{1 - (t-1)^2}, t); 0 \leq t \leq 2 \right\}.$$

It can be seen that  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_4)$ . From what previously said, these assumptions assure the existence of an expansion rate  $\alpha(T) > 0$ . In order to calculate this expansion rate, first we note that  $(x, y) \in T$  implies  $x_3 = y_3$ . If  $x_3 = 0$ , then process  $(x, y)$  has the form  $\lambda(1, 1, 0; 3, 2, 0)$ , with  $\lambda > 0$  and hence  $\alpha(x, y) = \min \{3/2, 2/1\} = 2$ . If  $x_3 \neq 0$ , then, as  $0 \leq \sqrt{1 - (t-1)^2} \leq 1$  for  $0 \leq t \leq 2$ , we get  $(x, y) = 1$ . It results therefore  $\alpha(T) = 2$  and the von Neumann processes are given by  $\lambda(1, 1, 0; 3, 2, 0)$ ,  $\lambda > 0$ . Omitting the multiplier  $\lambda$ , we consider a unique von Neumann process  $(x, y) = (1, 1, 0; 3, 2, 0)$ . With respect to this process, a von Neumann equilibrium solution consists in the existence of a price vector  $\bar{p} = (\bar{p}_1, \bar{p}_2, \bar{p}_3) > [0]$  which satisfies the relations

$$\bar{p}\bar{y} = 2\bar{p}\bar{x}, \quad (10)$$

$$\bar{p}y \leq 2\bar{p}x, \quad \forall (x, y) \in T, \quad (11)$$

$$\bar{p}\bar{y} > 0. \quad (12)$$

In a more explicit way, relations (10) and (12) are rewritten as:  $3\bar{p}_1 + 2\bar{p}_2 = 2\bar{p}_1 + 2\bar{p}_2$ ,  $3\bar{p}_1 + 2\bar{p}_2 > 0$ , from which  $\bar{p}_1 = 0$ ,  $\bar{p}_2 > 0$ . Therefore, the price vector which verifies (10) and (12) and which may form a von Neumann equilibrium, must be of the form  $\bar{p} = (0, \bar{p}_2, \bar{p}_3)$ , with  $\bar{p}_2 > 0$ . As a vector  $\bar{p}$  satisfies (10), (11) and (12) if and only if  $\bar{p}/(\bar{p}_1 + \bar{p}_2 + \bar{p}_3)$  satisfies the same relations, we limit ourselves to consider the following family of price vectors

$$P = \{(0, q, 1 - q) : 0 < q \leq 1\}.$$

The proof of the non-existence of a von Neumann equilibrium solution for technology  $T$ , results simplified by proving that for every vector  $\bar{p} \in P$  there exists a process  $(x, y) \in T_0 \subset T$  for which inequality (11) is not satisfied, that is  $\bar{p}y > 2\bar{p}x$ .

Let  $\bar{p} = (0, q, 1 - q)$ ,  $(1/2) < q \leq 1$ . By choosing the process  $(x, y) = (1, 1, 1; 3, 3, 1) \in T_0$  we get  $\bar{p}y = 3q + 1 - q = 2q + 1 > 2$ ,  $\bar{p}x = q + 1 - q = 1$ , that is  $\bar{p}y > 2\bar{p}x$ . Let now be  $\bar{p} = (0, q, 1 - q)$ ,  $0 < q \leq (1/2)$ . By choosing the process  $(x, y) = (1, 1, q^2; 3, 2 + \sqrt{1 - (q^2 - 1)^2}, q^2) \in T_0$ , we get  $\bar{p}y = q(2 + \sqrt{1 - (q^2 - 1)^2}) + (1 - q)q^2 = 2q + q^2\sqrt{2 - q^2} + q^2(1 - q)$ ,  $\bar{p}x = q + (1 - q)q^2$ .

The condition  $0 < q \leq (1/2)$  implies  $\sqrt{2 - q^2} > 1 - q$ , therefore in this case it holds  $\bar{p}y > 2\bar{p}x$ : relation (11) is not satisfied and hence a von Neumann equilibrium does not exist for the model taken into consideration.

The following result gives sufficient conditions for the existence of a von Neumann equilibrium.

**Theorem 1.** Let the technology  $T$  satisfy assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_4)$ ; moreover, assume that

- a) either  $T$  is a polyhedral cone;
- b) or  $T$  contains a von Neumann process  $(\tilde{x}, \tilde{y})$ , with  $\tilde{y} > [0]$ .

Then there exists a von Neumann equilibrium solution.

**Proof.** The existence of an equilibrium solution for the case a), i. e. for the classical von Neumann model, has been proved, e. g. by Kemeny, Morgenstern and Thompson (1956), Gale (1972), Howe (1960), Los (1971) and will not be repeated here.

Under assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_4)$  and under condition b), the existence of a von Neumann equilibrium solution is an immediate consequence of Lemma 2. □

**Remark 5.** A direct and self-contained proof of Theorem 1 (proof which does not make reference to other proofs for the polyhedral case, nor to Lemma 2) is provided by Makarov and Rubinov (1970, 1977). Note that condition b) of Theorem 1 may be viewed as a Slater-type regularity condition. Soyster (1974) proved Theorem 1 under a regularity condition weaker than b).

## 4. Growth Path, Balanced Growth Path, Maximal Balanced Growth Path.

Throughout the present Section we assume that the technology  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_3)$ . Suppose now that at a given period  $t$ ,  $t = 0, 1, 2, \dots$ , there are  $x_i(t) \geq 0$  units of the  $i$ -th good in the economy as a whole. We denote by  $x(t)$  the vector which represents the state of the economy at period  $t$ . The economy reaches another state  $x(t+1)$  at the next period  $(t+1)$ , through a production process described by the pair  $(x(t), x(t+1)) \in T$ . Similarly, by means of the inputs described by  $x(t+1)$ , the economic system can arrive at the state  $x(t+2)$ , through a production process described by the pair  $(x(t+1), x(t+2))$ , and so on. In this way, starting from  $x(0)$ , we can define a (feasible) growth path or (feasible) growth trajectory.

**Definition 4.** Given  $x^0 = x(0)$ , a finite sequence of vectors  $\{x(t)\}_{t=0}^N$  which satisfies the condition

$$(x(t), x(t+1)) \in T, \quad t = 0, 1, \dots, N-1, \quad x(0) = x^0$$

is called a *feasible path* or *feasible trajectory of (finite) horizon  $N$* , starting from  $x^0$ .

**Definition 5.** Given  $x^0 = (x(0))$ , an infinite sequence of vectors  $\{x(t)\}_{t=0}^\infty$  which satisfies the condition

$$(x(t), x(t+1)) \in T, \quad t = 0, 1, \dots, \quad x(0) = x^0$$

is called a *feasible path* or *feasible trajectory of infinite horizon*  $N$ , starting from  $x^0$ .

A feasible path  $\{x(t)\}_{t=0}^N$  represents a dynamic movement over  $N$  periods. In general, given an input vector  $x \geq [0]$ , the set  $\{y : (x, y)\}$  need not be a set consisting of a single element. In other words, starting from a given input vector, there may be several feasible growth paths. Let us denote by  $X_T(x^0, N)$  the set of all feasible paths of horizon  $N$  starting at a given initial vector  $x^0 \geq [0]$ , and by  $X_T(x^0)$  the set of all feasible paths of infinite horizon, always starting at a given initial vector  $x^0 \geq [0]$ . From assumption  $(T_3)$  it results that for every  $x^0 \in \mathbb{R}_+^n$  and every integer number  $N$ , the sets  $X_T(x^0, N)$  and  $X_T(x^0)$  are nonempty. Similarly, under assumptions  $(T_1)$  and  $(T_2)$ , we deduce that the unique trajectory (finite or infinite) which starts from the origin is the constant sequence of elements  $x(t) = [0]$ . Similarly to what appears in the literature concerning the classical von Neumann model, let us now give the notion of balanced growth path for the Gale- von Neumann model.

**Definition 6.** A feasible path  $\{x(t)\}_{t=0}^\infty$ , not identically the zero vector, is called a *balanced growth path* or *proportional growth path* (or *trajectory*) if

$$x(t+1) = \alpha x(t), \quad t = 0, 1, \dots,$$

where  $\alpha$  is a nonnegative constant.

The constant  $\alpha$  is termed the growth rate of the path (or growth factor; in this case the growth rate is  $\alpha - 1$ ). It is evident that a balanced growth path can be put in the form

$$x(t) = \alpha^t x, \quad x(0) = x \geq [0] \tag{13}$$

where the pair  $(x, \alpha)$  satisfies the relation

$$(x, \alpha x) \in T. \tag{14}$$

Conversely, every nonnegative number  $\alpha$  and every semipositive vector  $x$  satisfying (14), determine, thanks to (13), a balanced growth path. In Section 2 we have defined the growth rate  $\alpha(T)$  of a technology  $T$  which satisfies assumptions  $(T_1)$  and  $(T_2)$ , as the number

$$\alpha(T) = \max \{ \alpha : \alpha x \leq y, (x, y) \in T, (x, y) \neq ([0], [0]) \}.$$

**Definition 7.** A non trivial feasible growth path  $\{x(t)\}_{t=0}^\infty$  is called a *maximal balanced growth path* or *von Neumann balanced growth path* if

$$x(t+1) = \alpha(T)x(t), \quad t = 0, 1, \dots$$

From (13) and (14) we can note that a technology  $T$  possesses a maximal balanced growth path

$$x(t+1) = \alpha(T)x, \quad x(0) = x \geq [0] \quad (15)$$

if and only if

$$(x, \alpha(T)x) \in T. \quad (16)$$

We are now interested in the problem of the existence of a maximal balanced growth path, i. e. of a vector  $x \geq [0]$  such that (16) holds. A general theorem on the existence of the said path has been given by Nikaido (1968). We follow a proof different from the one of Nikaido and taken from Cruceanu (1978).

**Theorem 2.** If the technology  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_3)$ , there exists a maximal balanced growth path.

In order to prove the above theorem, we need some preliminaries definitions and results.

**Definition 8.** A convex cone  $K \subset \mathbb{R}^n$  is said to be *acute* if  $K \cap (-K) = \{[0]\}$ .

**Lemma 3.** If  $K$  is a closed acute convex cone, then there exists a vector  $p \in \mathbb{R}^n$  such that

$$pu > 0, \quad \forall u \in K, \quad u \neq [0]. \quad (17)$$

**Proof.** Let  $K^*$  be the polar cone of  $K$  (i. e.  $K^* = \{y : yx \geq 0, \forall x \in K\}$ ). We prove that  $K^*$  has a nonempty interior; suppose the contrary: hence the cone  $K^*$  is contained in a hyperplane  $H$ .

Let  $q \neq [0]$  be perpendicular to the said hyperplane:

$$H = \{u : qu = 0\}.$$

As  $K^* \subset H$ , for each  $v \in K^*$  we have  $qv = q(-v) = 0$ . Therefore  $q$  and  $-q$  belong to the cone  $K^{**}$ . Being  $K$  closed and convex, it holds  $K^{**} = K$ . So,  $q$  and  $-q$  belong to  $K$ , which is in contradiction with the assumption that  $K$  is an acute cone. Let us choose a vector  $p$  in the interior of  $K^*$ : there will exist a number  $\epsilon > 0$  such that  $p + r \in K^*$  for each vector  $r$ , with  $\|r\| \leq \epsilon$ . Let  $u \in K, u \neq [0]$ . By denoting  $r = -\epsilon u / 2 \|u\|$ , we obtain  $p + r \in K^*$ . Therefore  $(p + r)u \geq 0$ . It results

$$(p + r)u = pu - \frac{\epsilon}{2} \|u\|^2 \geq 0,$$

that is

$$pu \geq \frac{\epsilon}{2} \|u\|^2 > 0.$$

□

**Definition 9.** Let  $K \subset \mathbb{R}^n$  be a convex cone and let be given the set-valued map  $\Gamma : K \rightarrow 2^K$ , such that  $\Gamma(u) \neq \emptyset, \forall u \in K$ . The map  $\Gamma$  is said to be *sublinear* if

$$\Gamma(\lambda u + \lambda' u') \supset \lambda \Gamma(u) + \lambda' \Gamma(u'), \quad \forall u, u' \in K \text{ and } \forall \lambda, \lambda' \in \mathbb{R}_+. \quad (18)$$

The map  $\Gamma$  is said to be *closed* if its *graph*  $\{(u, v) : v \in \Gamma(u)\}$  is a closed set in  $\mathbb{R}^n \times \mathbb{R}^n$ .

From Definition 9 it results at once that the sublinear map  $\Gamma$  satisfies the equality

$$\Gamma(\lambda u) = \lambda \Gamma(u), \quad \forall u \in K, \quad \forall \lambda \in \mathbb{R}_+, \quad \lambda \neq 0.$$

Indeed, from (18) we get

$$\Gamma(\lambda u) \supset \lambda \Gamma(u) = \lambda \Gamma[(1/\lambda)\lambda u] \supset \Gamma(\lambda u).$$

On the other hand, for each  $u \in K$ , the set  $\Gamma(u)$  is convex: from (18), for  $v, v' \in \Gamma(u)$  and  $0 < \lambda < 1$ , we get

$$\lambda v + (1 - \lambda)v' \in \lambda \Gamma(u) + (1 - \lambda)\Gamma(u) \subset \Gamma[\lambda u + (1 - \lambda)u] = \Gamma(u).$$

The following result is interesting, not only for the subsequent proof of Theorem 2, but also as an autonomous result on the existence of characteristic vectors for a closed and sublinear map.

**Lemma 4.** Let  $K \subset \mathbb{R}^n, K \neq \{[0]\}$ , an acute closed convex cone and let  $\Gamma : K \rightarrow 2^K$  a closed sublinear map. Then, there exists vectors  $u, v \in K, u + v \neq [0]$ , and nonnegative numbers  $\lambda$  and  $\mu, \lambda + \mu \neq 0$ , such that  $v \in \Gamma(u)$  and  $\lambda u = \mu v$ .

**Proof.** Let us fix a point  $u^0 \in K, u^0 \neq [0]$ . Let us denote by  $F$  the hyperplane

$$F = \{u : pu = 0, p \neq [0]\}$$

where  $p$  is the vector which appears in Lemma 3. Let be

$$S = K \cap \{u^0 + F\}.$$

Obviously,  $S$  is closed and convex. We now prove that it is also bounded. Assume on the contrary that there exists in  $S$  a sequence  $\{u^n\}_{n=1}^\infty$  such that  $\|u^n\| \rightarrow \infty$  when  $n \rightarrow \infty$ . Every element of this sequence has the form  $u^n = u^0 + z^n$ , with  $z^n \in F$ . As  $\|z^n\| \rightarrow \infty$  when  $n \rightarrow \infty$ , we can suppose  $\|z^n\| \neq 0$  for every  $n$ . The sequence  $\{v^n\}_{n=1}^\infty, v^n = (u^0 + z^n) / \|z^n\|$  belongs to  $K$  and is bounded. Let  $\bar{v}$  be a limit point of this sequence. Obviously  $\bar{v}$  is also a limit point of the sequence  $z^n / \|z^n\|$ . Therefore  $\|\bar{v}\| = 1$ . From the closedness of  $K$  and  $F$  it results

$\bar{v} \in K, \bar{v} \in F$ . So, the cone  $K$  contains the element  $\bar{v} \neq [0]$  such that  $p\bar{v} = 0$ , in contradiction with (17).

We remark that if  $u \in K, u \neq [0]$ , then for  $\lambda = pu^0/pu$  we have  $\lambda u \in S$ . Let us consider the map  $G : S \rightarrow 2^S$  defined as follows.

$$G(u) = \left[ \bigcup_{\lambda > 0} \Gamma(\lambda u) \right] \cap S. \quad (19)$$

Without loss of generality, we can suppose that  $u \neq [0]$  implies  $[0] \notin \Gamma(u)$ . Indeed, if for  $\bar{u} \neq [0]$  we have  $[0] \in \Gamma(\bar{u})$ , then  $u = \bar{u}, v = [0], \lambda = 1, \mu = 0$  satisfy the conclusions of the lemma. So, we can suppose  $G(u) \neq \emptyset$  for every  $u \in S$ . We can see that the map  $G$  satisfies the assumptions of the *Kakutani fixed point theorem* (see, e. g., Nikaido (1968)). We already know that  $S$  is a convex compact set. From (19) it is easy to see that for each  $u \in S, G(u)$  is a convex set. It remains to prove that the map  $G$  is closed. Let be the sequence  $\{(u^n, v^n)\}_{n=1}^\infty$ , such that  $v^n \in G(u^n)$  and  $(u^n, v^n) \rightarrow (u, v)$  when  $n \rightarrow \infty$ . From (19) we deduce the existence of a positive sequence  $\{\lambda_n\}_{n=1}^\infty$  such that

$$v^n \in \Gamma(\lambda_n u^n), \quad n = 1, 2, \dots \quad (20)$$

From this we obtain the boundedness of the sequence  $\{\lambda_n\}_{n=1}^\infty$ . Indeed, if absurdly this sequence would have  $+\infty$  as a limit point, from the relation  $v^n/\lambda_n \in \Gamma(u^n)$ , it follows the closedness of the map  $\Gamma$ , in contradiction with  $[0] \notin \Gamma(u), u \neq [0]$ .

Let therefore be  $0 \leq \bar{\lambda} < \infty$  a limit point of the sequence  $\{\lambda_n\}_{n=1}^\infty$ . By (20) and the closedness of  $\Gamma$  we obtain

$$v \in \Gamma(\bar{\lambda}, u). \quad (21)$$

Being the set  $S$  closed, it results  $u, v \in S$ . Therefore  $v \in G(u)$ , that is the map  $G$  is closed. Therefore all the assumptions of the Kakutani fixed point theorem are satisfied:  $S$  is a convex and compact set, for each  $u \in S$  the set  $G(u)$  is convex,  $G$  is a closed map. Therefore the map  $G$  admits a fixed point. There exists a vector  $v \in S$  such that  $v \in G(v)$ . From this and from the definition of  $G$  by (19), we obtain the existence of a number  $\lambda > 0$  such that  $v \in \Gamma(\lambda u)$ .  $\square$

**Proof of Theorem 2.** Let us consider the set

$$Z = \{(x, y) : \alpha(T)x \leq y, (x, y) \in T\}.$$

As already remarked,  $Z$  is a closed convex cone in  $\mathbb{R}^n \times \mathbb{R}^n, Z \neq \{[0], [0]\}$ . Moreover, as this cone belongs to  $\mathbb{R}_+^n \times \mathbb{R}_+^n$ , this cone is acute. For  $(x, y) \in Z$  we put  $z = x + y - (1 + \alpha(T))x$ . From the definition of  $Z$  we have  $z \geq [0]$ . By assumption  $(T_3)$  there exists  $r \in \mathbb{R}_+^n$  such that  $(z, r) \in T$ . We have

$$(x + y, (1 + \alpha(T))y + r) = (1 + \alpha(T))(x, y) + (z, r) \in T.$$

Moreover, the following inequality holds

$$\alpha(T)(x + y) \leq (1 + \alpha(T))y + r.$$

Hence

$$((x + y, (1 + \alpha(T))y + r) \in Z.$$

Let us consider the following set-valued map  $\Omega : Z \rightarrow 2^Z$  :

$$\Omega(x, y) = \{(x + y, w) : (x + y, w) \in Z\}. \quad (22)$$

We have already seen that  $\Omega(x, y) \neq [0]$  for every  $(x, y) \in Z$ . The map  $\Omega$  is sublinear. Let  $(x^1, y^1), (x^2, y^2) \in Z$ ,  $(x^1 + y^1, w^1) \in \Omega(x^1, y^1)$ ,  $(x^2 + y^2, w^2) \in \Omega(x^2, y^2)$  and let  $\lambda_1$  and  $\lambda_2$  be nonnegative numbers. Being  $Z$  a convex cone, we get:

$$((\lambda_1(x^1 + y^1) + \lambda_2(x^2 + y^2)), \lambda_1 w^1 + \lambda_2 w^2) = \lambda_1(x^1 + y^1, w^1) + \lambda_2(x^2 + y^2, w^2) \in Z,$$

that is

$$\lambda_1(x^1 + y^1, w^1) + \lambda_2(x^2 + y^2, w^2) \in \Omega[\lambda_1(x^1 + y^1) + \lambda_2(x^2 + y^2)].$$

Hence

$$\Omega[\lambda_1(x^1 + y^1) + \lambda_2(x^2 + y^2)] \supset \lambda_1 \Omega(x^1, y^1) + \lambda_2 \Omega(x^2, y^2).$$

The map  $\Omega$  is closed. Consider the sequences  $\{(x^n, y^n)\}_{n=1}^\infty \subset Z$ ,  $\{w^n\}_{n=1}^\infty \subset \mathbb{R}_+$ ,  $(x^n + y^n, w^n) \in Z$ ,  $n = 1, 2, \dots$ . In other words,  $(x^n + y^n, w^n) \in \Omega(x^n, y^n)$ ,  $n = 1, 2, \dots$ . Suppose that  $(x^n, y^n) \rightarrow (\bar{x}, \bar{y})$ ,  $w^n \rightarrow \bar{w}$  when  $n \rightarrow \infty$ . As the cone  $Z$  is closed, we get  $(\bar{x}, \bar{y}) \in Z$ ,  $(\bar{x} + \bar{y}, \bar{w}) \in Z$ , that is  $(\bar{x} + \bar{y}, \bar{w}) \in \Omega(\bar{x}, \bar{y})$ . Therefore  $\Omega$  is a closed map. By Lemma 4, there exist the pairs  $(x', y'), (x'', y'') \in Z$ , at least one of them different from  $([0], [0])$ , and nonnegative numbers  $\lambda$  and  $\mu$ ,  $\lambda + \mu \neq 0$ , such that

$$\lambda(x', y') = \mu(x'', y'') \quad (23)$$

$$(x'', y'') \in \Omega(x', y'). \quad (24)$$

From (22) and (24) we obtain that there exists a vector  $w'$  such that

$$(x'', y'') = (x' + y', w'). \quad (25)$$

Relations (23) and (25) give

$$\lambda x' = \mu(x' + y'). \quad (26)$$

Equality (25) and assumption  $(T_2)$  show that  $x' \neq [0]$ ,  $x'' \neq [0]$ . From this and (26) we obtain  $\lambda \neq 0$ ,  $\mu \neq 0$ . Finally, we obtain

$$y' = \frac{\lambda - \mu}{\mu} x'. \quad (27)$$

Being  $(x', y')$  a von Neumann process, equality (27) implies

$$\frac{\lambda - \mu}{\mu} = \alpha(T).$$

□

**Remark 6.** A short proof of Theorem 2 has been given by Karlin (1959), but under assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_5)$ . On the other hand, we have already remarked that  $(T_1)$  and  $(T_5)$  imply  $(T_3)$ . Another short proof of Theorem 2 has been given by Furuya and Inada (1962), but under a different set of assumptions. More precisely, these authors assume  $(T_1)$ , but without convexity,  $(T_2)$  and  $(T_3)$ . Then they assume *super-additivity*:

**Assumption  $(T_6)$**  (Super-additivity). If  $(x^1, y^1) \in T$  and  $(x^2, y^2) \in T$ , there exists

$$(\lambda x^1 + (1 - \lambda)x^2, y) \in T \text{ such that } y \geq \lambda y^1 + (1 - \lambda)y^2, \text{ where } 0 \leq \lambda \leq 1.$$

Finally, they assume *primitivity*. The technological set  $T$  is called *primitive* if any semipositive input vector can yield a strictly positive output vector, after a finite number of periods. Formally:

**Assumption  $(T_7)$**  (Primitivity). There is a finite number  $s$  such that, for any  $x(0) \geq [0]$  and with  $x(s) > [0]$ ,  $(x(0), x(s)) \in T^{(s)}$ , being

$$T^{(s)} = \{(x(0), x(s)) : x(t)_{t=0}^s \in X_T(x^0, s)\}.$$

**Definition 10.** The ray  $\{\lambda x\}$ ,  $\lambda \geq 0$ , generated by a process of balanced maximal growth  $(x, \alpha(T)x)$ ,  $x \geq [0]$ , is called a *maximal balanced growth ray* or *von Neumann ray*.

On the grounds of Lemma 2, there exists a system of price vectors  $\bar{p} \geq [0]$  (von Neumann price vectors) such that

$$\bar{p}y - \alpha(T)\bar{p}x \leq 0 \text{ for any } (x, y) \in T. \quad (28)$$

For every process  $(x, y) \in T$ , let us consider the value

$$\pi(x, y; \bar{p}, \alpha(T)) = \bar{p}y - \alpha(T)\bar{p}x.$$

The number  $\pi(x, y; \bar{p}, \alpha(T))$  represents the revenue yielded by process  $(x, y)$ , when evaluated with the price vector  $\bar{p}$  and with the interest rate  $\alpha(T)$ . If  $(\bar{x}, \alpha(T)\bar{x})$  is a process with maximal balanced growth rate, it will hold

$$\pi(\bar{x}, \alpha(T)\bar{x}; \bar{p}, \alpha(T)) = \bar{p}\alpha(T)\bar{x} - \alpha(T)\bar{p}\bar{x} = 0. \quad (29)$$

**Definition 11.** Let  $(x, \alpha x) \in T$ ,  $\alpha \geq 0$ ,  $x \geq [0]$  be a process of balanced growth. We say that this process is *sustained by a price vector*  $p \geq [0]$  if

$$\pi(x, y; p, \alpha) = py - \alpha px \leq 0, \text{ for any } (x, y) \in T.$$

Now we give the following variant of Lemma 2.

**Theorem 3.** Under assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_3)$ , the processes of maximal balanced growth are sustained by a system of price vectors  $\bar{p} \geq [0]$ , called von Neumann price vectors, that is, for every process of maximal balanced growth the revenue computed with these prices and with the interest rate  $\alpha(T)$ , is zero.

A natural question now arises: under which conditions a balanced growth process sustained by a certain price system, is a maximal balanced growth process? The following result answers the said question.

**Theorem 4.** Assume  $(T_1)$ ,  $(T_2)$  and  $(T_3)$  and let  $(\hat{x}, \alpha\hat{x}) \in T$ ,  $\alpha \geq 0$ ,  $\hat{x} \geq [0]$ , a balanced growth process sustained by a price system  $p$ . If  $p > [0]$ , then  $\alpha = \alpha(T)$ , that is the process  $(\hat{x}, \alpha\hat{x})$  is a maximal balanced growth process.

**Proof.** We have

$$\pi(x, y; p, \alpha) = py - \alpha px \leq 0, \text{ for any } (x, y) \in T. \quad (30)$$

Let  $(\bar{x}, \alpha(T)\bar{x}) \in T$  be a process of maximal balanced growth. By (30) we get

$$\pi(\bar{x}, \alpha(T)\bar{x}; p, \alpha) = p\alpha(T)\bar{x} - \alpha p\bar{x} \leq 0. \quad (31)$$

Therefore

$$(\alpha(T) - \alpha)p\bar{x} \leq 0. \quad (32)$$

But  $p > [0]$  and  $\bar{x} \geq [0]$  imply  $p\bar{x} > 0$ , so from (32) we get  $\alpha(T) \leq \alpha$ . On the other hand, from  $(\hat{x}, \alpha\hat{x}) \in T$  we deduce  $\alpha \leq \alpha(T)$  and hence we obtain the equality  $\alpha = \alpha(T)$ .  $\square$

The maximal balanced growth ray is in general not unique. We can obtain its uniqueness and other useful properties of the maximal balanced growth processes under special assumptions.

**Definition 12.** The technology  $T$  is called *strictly convex* (**Assumption  $(T_8)$** ) if for any two processes  $(x, y) \in T$  and  $(u, v) \in T$ , with  $x$  and  $u$  linearly independent vectors, and for any numbers  $\delta > 0, \gamma > 0, \delta + \gamma = 1$ , there is an output vector  $w$  such that

$$(\delta x + \gamma u, w) \in T, \quad w > \delta y + \gamma v.$$

**Theorem 5.** Assume  $(T_1)$ ,  $(T_2)$  and  $(T_3)$ . Further assume  $(T_8)$ , i. e. the technology is strictly convex. Then:

- a) The maximal balanced growth ray or von Neumann ray is unique.
- b) If  $n \geq 2$ , then  $\alpha(T) > 0$ .
- c) If  $\{\lambda\bar{x}\}$ ,  $\lambda > 0$ , is the von Neumann ray of the technology and if  $\bar{p}$  is a von Neumann price vector, then  $\bar{p}_i = 0$  implies  $\bar{x}_i > 0$ ,  $\bar{x}_j = 0$  for  $j \neq i$  (at most one component of  $\bar{p}$  can be zero).

**Proof.** Let  $\{\lambda\bar{x}\}$ ,  $\lambda > 0$ , a von Neumann ray and  $\bar{p}$  a von Neumann price vector. First we prove that for every process  $(x, y) \in T$  for which  $x$  is not proportional to  $\bar{x}$ , we have  $\pi(x, y; \bar{p}, \alpha(T)) < 0$ . Indeed, as  $(\bar{x}, \alpha(T)\bar{x}) \in T$ , from the strict convexity of the technology  $T$  it results the existence of a vector

$$w > \frac{y + \alpha(T)\bar{x}}{2} \quad (33)$$

such that

$$\left( \frac{x + \bar{x}}{2}, w \right) \in T. \quad (34)$$

Inequality (33) implies

$$\bar{p} \frac{y + \alpha(T)\bar{x}}{2} - \bar{p}w < 0, \quad (35)$$

and taking (28) and (34) into account, we obtain

$$\bar{p}w - \alpha(T)\bar{p} \frac{x + \bar{x}}{2} \leq 0. \quad (36)$$

Putting together (35) and (36) we obtain

$$\pi(x, y; \bar{p}, \alpha(T)) = \bar{p}y - \alpha(T)\bar{p}x < 0.$$

Now we prove points a), b) and c) of the theorem.

a) Let us absurdly suppose that the technology  $T$  generates a von Neumann ray  $\{\lambda x\}$  different from the ray  $\{\lambda\bar{x}\}$ . Hence,  $(x, \alpha(T)x)$  is a maximal balanced growth process and  $x$  is not proportional to  $\bar{x}$ . On the ground of what previously proved, it holds  $\pi(\bar{x}, \alpha(T)\bar{x}; \bar{p}, \alpha(T)) < 0$ , in contradiction with Theorem 3. Whence,  $x$  must be proportional to  $\bar{x}$ , which proves the desired uniqueness.

b) If  $n \geq 2$ , we can choose a vector  $x \geq [0]$  not proportional to  $\bar{x}$ . By assumption  $(T_3)$  there exists a vector  $y$  such that  $(x, y) \in T$ . It results

$$\pi(x, y; \bar{p}, \alpha(T)) = \bar{p}y - \alpha(T)\bar{p}x < 0,$$

whence  $0 \leq \bar{p}y < \alpha(T)\bar{p}x$ , which implies  $\alpha(T) > 0$ .

c) Let  $\bar{p}_i = 0$ . We choose a vector  $x$  such that  $x_i > 0$  and  $x_j = 0$  for  $j \neq i$ . From assumption  $(T_3)$  it results the existence of an output vector  $y$  such that  $(x, y) \in T$ . We obtain

$$\bar{p}y - \alpha(T)\bar{p}x = \bar{p}y \geq 0 \quad (37)$$

and taking inequality (28) into account,

$$\bar{p}y - \alpha(T)\bar{p}x \leq 0. \quad (38)$$

Therefore  $\bar{p}y - \alpha(T)\bar{p}x = 0$ , whence  $x$  must be proportional to  $\bar{x}$ , implying that  $\bar{x}_i > 0$ ,  $\bar{x}_j = 0$  for  $j \neq i$ . Now, if there were another vanishing component of  $\bar{p}$ , say, the  $k$ -th one ( $k \neq i$ ), the components of  $\bar{x}$  would also satisfy  $\bar{x}_k > 0$ ,  $\bar{x}_j = 0$  ( $j \neq k$ ) for the same reason as above, yielding a contradiction. □

**Remark 7.** The uniqueness of the von Neumann ray is obtained by Furuya and Inada (1962) under the assumptions mentioned in Remark 6 and adding a further assumption: these authors assume that  $T$  is *strongly super additive*, a property slightly weaker than strict convexity:

**Assumption  $(T_9)$**  (Strong super additivity). Let  $x^1 \neq [0]$ ,  $x^2 \neq [0]$  and  $x^1 \neq \alpha x^2$  for any positive  $\alpha$ . The cone  $T$  is said to be strongly super additive if for  $(x^1, y^1) \in T$ , and  $(x^2, y^2) \in T$ , there exists a process  $(x^1 + x^2, y^*) \in T$  such that  $y^* \geq y^1 + y^2$ .

We draw the reader's attention on the fact that Theorem 2 of Furuya and Inada (1962, page 100) is not fully correct. It has been corrected by Fisher (1963).

Assumption  $(T_9)$  and assumption  $(T_8)$  of strict convexity (Definition 12) have been criticized by Morishima (1964) who regards them as economically implausible properties. This author obtains strong results on the uniqueness of the von Neumann ray and on the von Neumann price vector, by means of various assumptions (in general weaker than strict convexity of  $T$  and than the assumption of strong super additivity  $(T_9)$ ), but imposing also an *indecomposability property*: let  $J(x, y) = \{j : x_j = 0 \text{ or } y_j = 0\}$ . The technological set  $T$  is called *indecomposable* at  $(x, y)$  if  $J(x, y)$  is empty or if  $J(x, y)$  contains at least one  $j$  such that  $x_j > 0$  or  $y_j > 0$ .

Morishima (1964, page 180) assumes then:

**Assumption  $(T_{10})$**  (Indecomposability). The technological set  $T$  is indecomposable at every point  $(x, y) \in T$ .

## 5. Optimal Final Paths

In the present section we discuss some criteria to compare paths (or trajectories). This will be done in terms of the final state. We suppose that the technological set  $T$  satisfies the assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_3)$ .

**Lemma 5.** Let us consider the set

$$B = \{y : (x, y) \in T, x \in A\},$$

where  $A$  is a nonempty set of  $\mathbb{R}_+^n$ . Then:

- a) If  $A$  is convex, then  $B$  is convex.
- b) If  $A$  is compact, then  $B$  is compact.

**Proof.**

a) Let the vectors  $y', y'' \in B$  and let us consider the number  $\lambda$ ,  $0 < \lambda < 1$ . Hence, there exist vectors  $x', x''$  such that  $(x', y'), (x'', y'') \in A$ . Being  $A$  and  $T$  both convex sets, we obtain  $\lambda x' + (1 - \lambda)x'' \in A$ ,  $(\lambda x' + (1 - \lambda)x'', \lambda y' + (1 - \lambda)y'') = \lambda(x', y') + (1 - \lambda)(x'', y'') \in T$ . Hence,  $\lambda y' + (1 - \lambda)y'' \in B$ .

b) Let us suppose absurdly that the set  $B$  is unbounded. Therefore, there exists a sequence  $\{(x^n, y^n)\}_{n=1}^\infty \subset T$  with  $x^n \in A$  for all  $n$ , such that  $\|y^n\| \rightarrow \infty$  when  $n \rightarrow \infty$ . The bounded sequence  $\{y^n / \|y^n\|\}_{n=1}^\infty$  contains a subsequence  $\{y^{n_k} / \|y^{n_k}\|\}_{k=1}^\infty$  convergent to an element  $y$ . As  $T$  is a cone,  $(x^{n_k} / \|y^{n_k}\|, y^{n_k} / \|y^{n_k}\|) \in T$ ,  $k = 1, 2, \dots$ . Taking the limit for  $k \rightarrow \infty$ , being  $A$  a bounded set and  $T$  a closed set, we obtain  $([0], y) \in T$ . But  $\|y\| = 1$ , in contrast with  $(T_2)$ . Hence, the set  $B$  is bounded. Now consider a sequence  $\{y^n\}_{n=1}^\infty \subset B$  such that  $y^n \rightarrow \bar{y}$  when  $n \rightarrow \infty$ . Hence, there exists a sequence  $\{x^n\}_{n=1}^\infty \subset A$  for which  $(x^n, y^n) \in T$ ,  $n = 1, 2, \dots$ . The sequence  $\{x^n\}_{n=1}^\infty$ , belonging to the set  $A$ , is bounded. Let  $\bar{x}$  be a limit point of this sequence. Being  $A$  a closed set, we have  $\bar{x} \in A$ , which, together with the closedness of  $T$  implies  $(\bar{x}, \bar{y}) \in T$ . Hence,  $\bar{y} \in B$ , that is  $B$  is a closed set. □

Consider now the vector  $x^0 \in \mathbb{R}_+^n$ . We recall that we have denoted by  $X_T(x^0, N)$  the set of all feasible paths (or trajectories) of horizon  $N$  starting at a given initial state  $x^0 \geq [0]$ . We consider the sequence of sets  $B_T(x^0, 1), B_T(x^0, 2), \dots$ , defined as follows:

$$B_T(x^0, 1) = \{y : (x^0, y) \in T\} \tag{39}$$

$$B_T(x^0, t + 1) = \{y : (x, y) \in T, x \in B_T(x^0, t)\}, t = 1, 2, \dots \tag{40}$$

The set  $B_T(x^0, t)$  therefore represents all the states that the economy can reach in  $t$  periods, starting from the initial situation  $x^0$ . By Lemma 5 every set  $B_T(x^0, t)$  is convex and compact.

**Lemma 6.** The set  $X_T(x^0, N)$  is convex and compact in the cartesian product

$$V = \underbrace{\mathbb{R}_+^n \times \mathbb{R}_+^n \times \dots \times \mathbb{R}_+^n}_{N+1 \text{ factors}}$$

**Proof.** Let  $\{x^1(t)\}_{t=0}^\infty, \{x^2(t)\}_{t=0}^\infty$  be two paths of  $X_T(x^0, N)$ . Therefore we have  $(x^1(t), x^1(t+1)) \in T, (x^2(t), x^2(t+1)) \in T, t = 0, 1, \dots, N-1, x^1(0) = x^2(0) = x^0$ . For  $\lambda \in (0, 1)$  let us consider the sequence  $\{x^\lambda(t)\}_{t=0}^N$ , with  $x^\lambda(t) = \lambda x^1(t) + (1-\lambda)x^2(t), t = 0, 1, \dots, N$ . Obviously,  $x^\lambda(0) = \lambda x^1(0) + (1-\lambda)x^2(0) = x^0$ , but from convexity of the cone  $T$  we obtain  $(x^\lambda(t), x^\lambda(t+1)) = \lambda(x^1(t), x^1(t+1)) + (1-\lambda)(x^2(t), x^2(t+1)) \in T$ , that is  $\{x^\lambda(t)\}_{t=0}^\infty = X_T(x^0, N)$ . As  $X_T(x^0, N) \subset \{x^0\} \times B_T(x^0, 1) \times \dots \times B_T(x^0, N)$ , the set  $X_T(x^0, N)$  is bounded.

It remains to prove that the same set is closed. Let be the sequence of paths  $\{x^n(t)\}_{t=0}^N \in X_T(x^0, N), n = 1, 2, \dots$ , such that  $x^n(t) \rightarrow \bar{x}(t)$  when  $n \rightarrow \infty$  for all  $t = 0, 1, \dots, N$ . We shall prove that  $\{\bar{x}(t)\}_{t=0}^N \in X_T(x^0, N)$ . From  $x^n(0) = x^0$  for all  $n$ , it results  $\bar{x}(0) = x^0$ . In the same way for all  $n$  we have  $(x^n(t), x^n(t+1)) \in T, t = 0, 1, \dots, N-1$ , so that, taking the limit for  $n \rightarrow \infty$ , being  $T$  closed, we obtain  $(\bar{x}(t), \bar{x}(t+1)) \in T, t = 0, 1, \dots, N-1$ , that is  $\{\bar{x}(t)\}_{t=0}^\infty \in X_T(x^0, N)$ .  $\square$

Now we introduce a *preference function* (or *utility function*)  $u(x)$  in terms of which it is possible to define an optimality criterion. Usually,  $u : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  is a continuous function and in some models other economic and/or mathematical properties are imposed (see, e. g., Radner (1961), Morishima (1964)). Important examples of preference functions are:

1)  $u(x) = \max \{c : x_i \geq c\alpha_i, i = 1, \dots, n\}$  where the coefficients  $\alpha_i$  are fixed nonnegative numbers with  $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ . In this case the numbers  $\alpha_1, \dots, \alpha_n$  represent the “desired proportions” of the several commodities.

2)  $u(x) = \sum_{i=1}^n p_i x_i = px$ , with  $p_i \geq 0, i = 1, \dots, n$ . We can interpret the component  $p_i$  of  $p$  as the price of the  $i$ -th commodity, so that  $u(x)$  gives the value of  $x$  under the set of prices  $p_1, \dots, p_n$ .

The growth path which maximizes the preference function with respect to the final state is the solution of the problem

$$\text{maximize } u(x(N)) \tag{41}$$

$$\text{subject to } \{x(t)\}_{t=0}^N \in X_T(x^0, N). \tag{42}$$

The existence of a solution of the said problem is an immediate consequence of Lemma 6 and of Weierstrass Theorem: the continuous function  $f(\{x(t)\}_{t=0}^N) = u(x(N))$  admits a maximum on the compact set  $X_T(x^0, N)$ .

**Definition 13.** Every optimal solution of problem (41)-(42) is called *final optimal path* or *final optimal trajectory* with respect to the preference function  $u(x)$ .

We limit our next treatment to the preference function  $u(x) = px$ , with  $p \in \mathbb{R}_+^n$ .

**Definition 14.** The final optimal path with respect to the preference function  $u(x) = px$  is called *p-optimal path* or *p-optimal trajectory*.

By means of some classical results on duality for a concave programming problem, we shall obtain necessary and sufficient conditions for  $p$ -optimality. We consider the following set, related to the technology  $T$  :

$$\tilde{T} = \{(\tilde{x}, \tilde{y}) : \text{there exists } (x, y) \in T \text{ with } x \leq \tilde{x}, [0] \leq \tilde{y} \leq y\}. \quad (43)$$

Obviously,  $T \subset \tilde{T}$ , moreover, from (43) it results at once that  $T$  satisfies assumption  $(T_5)$ .

**Lemma 7.** If  $T$  satisfies assumptions  $(T_1)$  and  $(T_2)$ , then also  $\tilde{T}$  satisfies these assumptions. If, moreover, we assume that  $T$  satisfies also  $(T_3)$ , then

$$\tilde{T} = \{(\tilde{x}, \tilde{y}) : \text{there exists } y \text{ such that } (\tilde{x}, y) \in T, [0] \leq \tilde{y} \leq y\}. \quad (44)$$

**Proof.** The fact that  $\tilde{T}$  is a convex cone which satisfies assumption  $(T_2)$  directly comes from (43) and from assumptions  $(T_1)$  and  $(T_2)$  made on the set  $T$ .

Now we prove that  $\tilde{T}$  is a closed set. Consider the sequence  $\{(\tilde{x}^n, \tilde{y}^n)\}_{n=1}^\infty \subset \tilde{T}$  such that  $(\tilde{x}^n, \tilde{y}^n) \rightarrow (\tilde{x}, \tilde{y})$  when  $n \rightarrow \infty$ .

We prove that  $(\tilde{x}, \tilde{y}) \in \tilde{T}$ . From (43) it results the existence of a sequence  $\{(x^n, y^n)\}_{n=1}^\infty \subset T$  such that  $x^n \leq \tilde{x}^n$ ,  $[0] \leq \tilde{y}^n \leq y^n$ , for each  $n$ . As the sequence  $\{\tilde{x}^n\}_{n=1}^\infty$  is convergent, it is bounded. The same holds also for the sequence  $\{x^n\}_{n=1}^\infty$ . From Lemma 5 we deduce the boundedness of the sequence  $\{y^n\}_{n=1}^\infty$ . Let  $(x, y)$  be a limit point of the sequence  $\{x^n, y^n\}_{n=1}^\infty$ . We obtain  $x \leq \tilde{x}$ ,  $[0] \leq \tilde{y} \leq y$ , that is  $(\tilde{x}, \tilde{y}) \in \tilde{T}$ .

It remains to prove that when the set  $T$  satisfies also assumption  $(T_3)$ , then  $\tilde{T}$  can be defined also by means of relation (44). It is sufficient to prove that, with  $(\tilde{x}, \tilde{y}) \in \tilde{T}$ , there exists a vector  $y$ , with  $(\tilde{x}, y) \in T$  and  $[0] \leq \tilde{y} \leq y$ . Let therefore be  $(x, \tilde{y}) \in \tilde{T}$ . From (43) we deduce the existence of a process  $(v, w) \in T$  which verifies the inequalities  $v \leq \tilde{x}$ ,  $[0] \leq \tilde{y} \leq w$ . As  $\tilde{x} - v \geq [0]$ , from assumption  $(T_3)$ , there exists a vector  $z$  such that  $(\tilde{x} - v, z) \in T$ . By denoting  $y = z + w$  we obtain  $[0] \leq \tilde{y} \leq y$  and  $(\tilde{x}, y) = (\tilde{x} - v, z) + (v, w) \in T$ . □

Let  $X_{\tilde{T}}(x^0, N)$  be the set of the growth trajectories of horizon  $N$  from  $x^0$ , referred to the technology set  $\tilde{T}$ . From Lemmas 6 and 7 we deduce that  $X_{\tilde{T}}(x^0, N)$  is a convex and compact subset of  $V$ . Similarly, from the inclusion  $T \subset \tilde{T}$  we deduce the inclusion  $X_T(x^0, N) \subset X_{\tilde{T}}(x^0, N)$ .

**Lemma 8.** Let us suppose that the technology set  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_3)$ . Let be  $p \in \mathbb{R}^n$ ,  $p \geq [0]$ . A trajectory  $\{x^*(t)\}_{t=0}^\infty$ ,  $p$ -optimal in  $X_T(x^0, N)$  is also  $p$ -optimal in  $X_{\tilde{T}}(x^0, N)$ .

**Proof.** Let us absurdly consider that there exists a trajectory  $\{y(t)\}_{t=0}^N \in X_{\tilde{T}}(x^0, N)$  such that  $p y(N) > p x^*(N)$ . Hence we have  $(x^0, y(1)) \in \tilde{T}$ ,  $(y(1), y(2)) \in \tilde{T}$ , ...,  $(y(N-1), y(N)) \in \tilde{T}$ . From (44) it results the existence of vectors  $z(1), z(2), \dots, z(N)$  for which  $(x^0, z(1)) \in T$ ,

$(y(1), z(2)) \in T, \dots, (y(N-1), z(N)) \in T$  and  $y(1) \leq z(1), y(2) \leq z(2), \dots, y(N) \leq z(N)$ . As  $z(1) - y(1) \geq [0]$ , there exists a vector  $v(1) \geq [0]$  such that  $(z(1) - y(1), v(1)) \in T$ . We get

$$(z(1), z(2) + v(1)) = (y(1), z(2)) + (z(1) - y(1), v(1)) \in T.$$

As  $z(2) + v(1) - y(2) \geq [0]$ , there exists a vector  $v(2) \geq [0]$ , such that  $(z(2) + v(1) - y(2), v(2)) \in T$ . We obtain  $(z(2) + v(1), z(3) + v(2)) = (y(2), z(3)) + (z(2) + v(1) - y(2), v(2)) \in T$ .

Going on in this way, we can build nonnegative vectors  $v(1), v(2), \dots, v(N)$  such that  $(x^0, z(1)) \in T, (z(1), z(2) + v(1)) \in T, (z(2) + v(1), z(3) + v(2)) \in T, \dots, (z(t-1) + v(t-2), z(t) + v(t-1)) \in T, \dots, (z(N-1) + v(N-2), z(N) + v(N-1)) \in T$ .

Therefore, the sequence  $x^0, z(1), z(2) + v(1), \dots, z(t) + v(t-1), \dots, z(N) + v(N-1)$  forms a trajectory of  $X_T(x^0, N)$ . By means of the inequality  $z(N) \geq y(N)$ , we obtain  $p(z(N) + v(N-1)) = pz(N) + pv(N-1) \geq py(N) > px^*(N)$ , in contradiction with the  $p$ -optimality of the trajectory  $\{x^*(t)\}_{t=0}^N$  in  $X_T(x^0, N)$ . □

**Remark 8.** With a slight modification of the previous proof we can get the following more general version of Lemma 8:

Let  $T$  satisfy assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_3)$  and let the preference function  $u$  be non decreasing, i. e.  $x^1 \geq x^2$  implies  $u(x^1) \geq u(x^2)$ ; then any final optimal trajectory  $\{x^*(t)\}_{t=0}^N$  with respect to  $u$  in  $X_T(x, N)$  is also final optimal with respect to  $u$  in  $X_{\bar{T}}(x^0, N)$ .

For every  $x \in \mathbb{R}_+^n$  and every index  $0 \leq t \leq N-1$ , let us consider the problem

$$\text{maximize } px(N) \tag{45}$$

$$\text{subject to } (x(\tau), x(\tau+1)) \in T, \tau = t, t+1, \dots, N-1, \tag{46}$$

$$x(t) = x. \tag{47}$$

We denote by  $V_N^t(x)$  the optimal value of this problem. From Lemma 8 we deduce that  $V_N^t(x)$  is at the same time also optimal value of the problem

$$\text{maximize } px(N) \tag{48}$$

$$\text{subject to } (x(\tau), x(\tau+1)) \in \tilde{T}, \tau = t, t+1, \dots, N-1, \tag{49}$$

$$x(t) = x. \tag{50}$$

The value  $V_N^t(x)$  represents the maximum value (with respect to the price system  $p$ ) of a final state, that it is possible to obtain after  $N-t$  periods, starting from the initial state  $x$ .

So, for each index  $0 \leq t \leq N-1$ , and for each vector  $x \in \mathbb{R}_+^n$  of the value  $V_N^t(x)$  we have defined a function  $V_N^t : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ . The main properties of  $V_N^t$  are described in the following lemma.

**Lemma 9.** The function  $V_N^t$  has the following properties:

- a) It is a concave, positively homogeneous and upper semicontinuous function.
- b)  $V_N^t = \max_{(x,y) \in T} V_N^{t+1}(y) = \max_{(x,y) \in \tilde{T}} V_N^{t+1}(y)$ ,  $t = 0, 1, \dots, N-1$ .
- c) If  $\{x^*(t)\}_{t=0}^N$  is a  $p$ -optimal growth trajectory (independent from the initial state), then

$$V_N^0(x^*(0)) = \dots = V_N^t(x^*(t)) = \dots = V_N^{N-1}(x^*(N-1)) = p x^*(N).$$

**Proof.** For every  $x \in \mathbb{R}_+^n$  and every index  $0 \leq t \leq N-1$  we denote the following set

$$X_T(x, N, t) = \left\{ \{x^*(\tau)\}_{\tau=t}^N : x(t) = x, (x(\tau), x(\tau+1)) \in T, \tau = t, \dots, N-1 \right\}.$$

On the ground of this notation we can write

$$V_N^t(x) = \max \left\{ p x(N) : \{x(\tau)\}_{\tau=t}^N \in X_T(x, N, t) \right\}.$$

- a) Let  $x', x'' \in \mathbb{R}_+^n$ . Let us choose the trajectories  $\{x'(\tau)\}_{\tau=t}^N \in X_T(x', N, t)$ ,  $\{x''(\tau)\}_{\tau=t}^N \in X_T(x'', N, t)$ , such that  $V_N^t(x') = p x'(N)$ ,  $V_N^t(x'') = p x''(N)$ .

For  $\lambda \in (0, 1)$  let us denote  $x^\lambda(\tau) = \lambda x'(\tau) + (1-\lambda)x''(\tau)$ ,  $\tau = t, t+1, \dots, N$ . Obviously we have  $\{x^\lambda(\tau)\}_{\tau=t}^N \in X_T(\lambda x' + (1-\lambda)x'', N, t)$ . Hence,  $V_N^t(\lambda x' + (1-\lambda)x'') \geq p x^\lambda(N) = p[\lambda x'(N) + (1-\lambda)x''(N)] = \lambda V_N^t(x') + (1-\lambda)V_N^t(x'')$ , therefore the function  $V_N^t$  is concave. Consider now  $x \in \mathbb{R}_+^n$  and the scalar  $\alpha > 0$ . Being  $T$  a cone, we get

$$X_T(\alpha x, N, t) = \alpha X_T(x, N, t),$$

from which we have

$$V_N^t(\alpha x) = \alpha V_N^t(x),$$

that is, the function  $V_N^t$  is positively homogeneous.

Let us consider the sequence  $\{x^n\}_{n=1}^\infty \subset \mathbb{R}_+^n$  and the vector  $x \in \mathbb{R}_+^n$  such that  $x^n \rightarrow x$  when  $n \rightarrow \infty$ . For every  $n$ , let us choose the trajectory  $\{x^n(\tau)\}_{\tau=t}^N \in X_T(x^n, N, t)$  for which  $V_N^t(x^n) = p x^n(N)$ . The sequence  $\{x^n\}_{n=1}^\infty$ , being convergent, is bounded. Hence, by Lemma 5, it results that are bounded also the sequences  $\{x^n(t+1)\}_{n=1}^\infty$ ,  $\{x^n(t+2)\}_{n=1}^\infty$ , ...,  $\{x^n(N)\}_{n=1}^\infty$ . By extraction of successive convergent subsequences, we obtain a sequence of indices  $\{n_k\}_{k=1}^\infty$  and vectors  $x(t+1), \dots, x(N)$ , such that  $\lim_{k \rightarrow \infty} x^{n_k}(\tau) = x(\tau)$ ,  $\tau = t+1, t+2, \dots, N$ .

By denoting  $x(t) = x$  and being  $T$  a closed set, it results

$$\{x(\tau)\}_{\tau=t}^N \in X_T(x, N, t).$$

Hence

$$V_N^t(x) \geq p x(N) = \lim_{k \rightarrow \infty} p x^{n_k}(N) = \lim_{k \rightarrow \infty} V_N^t(x^{n_k}),$$

which implies

$$V_N^t(x) \geq \limsup_{n \rightarrow \infty} V_N^t(x^n).$$

- b) Let  $\{x^*(\tau)\}_{\tau=t}^N \in X_T(x, N, t)$  such that  $V_N^t(x) = p x^*(N)$ .  
As  $\{x^*(\tau)\}_{\tau=t+1}^N \in X_T(x^*(t+1), N, t+1)$ , we obtain

$$V_N^{t+1}(x^*(t+1)) \geq p x^*(N).$$

From this relation and being  $(x, x^*(t+1)) \in T$ , it results

$$V_N^t(x) \leq \sup_{(x,y) \in T} V_N^{t+1}(y). \quad (51)$$

As the set  $\{x : (x, y) \in T\}$  is compact, the upper semicontinuous function  $V_N^{t+1}$  admits a maximum  $m$  on this set. Let  $(x, \bar{y}) \in T$  such that

$$\max_{(x,y) \in T} V_N^{t+1}(y) = V_N^{t+1}(\bar{y}).$$

Let us absurdly suppose that (51) holds only with the strict inequality relation, that is that

$$V_N^t(x) < V_N^{t+1}(\bar{y}). \quad (52)$$

Let us choose a trajectory  $\{x^{**}(\tau)\}_{\tau=t+1}^N \in X_T(\bar{y}, N, t+1)$  for which

$$V_N^{t+1}(\bar{y}) = p x^{**}(N). \quad (53)$$

By denoting

$$x(t) = x$$

$$x(\tau) = x^{**}(\tau), \quad \tau = t+1, \dots, N,$$

we obtain  $\{x(\tau)\}_{\tau=t}^N \in X_T(x, N, t)$ , and at the same time (52) and (53) imply  $V_N^t(x) < p x(N)$ , inequality which contradicts the definition of  $V_N^t$ . Hence relation (51) must hold with the equality sign. Finally, the equality

$$\max_{(x,y) \in T} V_N^t(y) = \max_{(x,y) \in \tilde{T}} V_N^t(y),$$

is an immediate consequence of Lemma 8.

- c) Let  $\{x^*(\tau)\}_{\tau=0}^N$  be a  $p$ -optimal trajectory of  $X_T(x^*(0), N)$ .

Let us note that from  $X_T(x^*(0), N) = X_T(x^*(0), N, 0)$  it results the equality

$$V_N^0(x^*(0)) = p x^*(N).$$

As  $\{x^*(\tau)\}_{\tau=1}^N \in X_T(x^*(1), N, 1)$ , we have

$$V_N^1(x^*(1)) \geq p x^*(N). \quad (54)$$

On the other hand

$$V_N^1(x^*(1)) \leq \max_{(x^*(0), y) \in T} V_N^1(y) = V_N^0(x^*(0)) = p x^*(N). \quad (55)$$

From (54) and (55) it results

$$V_N^1(x^*(1)) = p x^*(N). \quad (56)$$

By repeating this procedure we get

$$V_N^t(x^*(t)) = p x^*(N), \quad t = 2, 3, \dots, N - 1.$$

□

Now we formulate the following necessary and sufficient condition of  $p$ -optimality.

**Theorem 6.** Let us suppose that the technology  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$ ,  $(T_3)$  and  $(T_4)$ . Let  $p \in \mathbb{R}^n$ ,  $p \geq [0]$  be a price vector. If  $x^0 > [0]$ , then the trajectory  $\{x^*(t)\}_{t=0}^N$  is  $p$ -optimal in  $X_T(x^0, N)$  if and only if there exist vectors

$$p(t) \in \mathbb{R}^n, \quad p(t) \geq [0], \quad t = 0, 1, \dots, N,$$

such that:

- 1)  $p(0) x^*(0) = p(1) x^*(1) = \dots = p(N) x^*(N)$ ;
- 2)  $p(t) x - p(t+1) y \geq 0$ , for every  $(x, y) \in T$ ,  $t = 0, 1, \dots, N - 1$ ;
- 3)  $p(N) = p$ .

In order to prove the previous theorem we need some definitions and results of classical concave programming (see, e.g. Gale (1967), Geoffrion (1971), Rockafeller (1970)).

Let us consider the following concave maximization problem

$$(P) \quad \begin{aligned} & \text{maximize } u(x, y), \\ & (x, y) \in T \\ & x = \bar{z} \end{aligned}$$

when  $u(\cdot, \cdot)$  is a concave function and  $\bar{z}$  is a given input.

**Definition 15.** We say that the triplet  $(x^*, y^*, \lambda^*)$ ,  $x^* \in \mathbb{R}^n$ ,  $y^* \in \mathbb{R}^n$ ,  $\lambda^* \in \mathbb{R}^n$ , satisfy the optimality conditions for problem (P) if

- a)  $(x^*, y^*) \in T$ ,
- b)  $(x^*, y^*)$  maximizes the Lagrangian function  $u(x, y) - \lambda^*(x - \bar{z})$  on  $T$ ,

c)  $x^* = \bar{z}$ .

**Definition 16.** Let us suppose that problem (P) admits a solution. We say that the Kuhn-Tucker theorem holds for (P) if for each optimal solution  $(x^*, y^*)$  of the said problem, there exists at least one vector  $\lambda^* \in \mathbb{R}^n$ , called *optimal multipliers vector* or *Kuhn-Tucker price vector* or *Kuhn-Tucker multipliers vector*, such that the triplet  $(x^*, y^*, \lambda^*)$  satisfies the optimality conditions given in Definition 15.

Now let us consider the set

$$Z = \{x : \text{there exists } y \text{ such that } (x, y) \in T\}.$$

For every  $z \in \mathbb{R}^n$  let us consider the function

$$v(z) = \begin{cases} \sup \{u(x, y) / (x, y) \in T, x = z\} & \text{if } z \in Z. \\ -\infty, & \text{if } z \notin Z. \end{cases}$$

Function  $v(z)$  ( $v : \mathbb{R}^n \rightarrow [-\infty, +\infty]$ ) is called “perturbation function” for problem (P). It can be proved that the set  $Z$  is convex and that function  $v$  is concave on  $Z$ .

We note also that problem (P) admits feasible solutions if and only if  $\bar{z} \in Z$ . In any case the optimal value of (P) is just  $v(\bar{z})$ . We recall also that  $\lambda^* \in \mathbb{R}^n$  is a *subgradient* of the concave function  $v$  at the point  $z$ , if the following inequality is satisfied:

$$v(z) - v(\bar{z}) \leq \lambda^* (z - \bar{z}), \quad \forall z \in \mathbb{R}^n.$$

**Definition 17.** Problem (P) is said to be *stable* if the optimal value  $v(\bar{z})$  is finite and if function  $v$  admits a subgradient at  $\bar{z}$ .

**Theorem 7.** Let us suppose that problem (P) admits at least an optimal solution. The Kuhn-Tucker theorem holds for (P) if and only if (P) is stable. Moreover,  $\lambda^*$  is an optimal multipliers vector if and only if it is a subgradient of function  $v$  at the point  $\bar{z}$ .

In general, given a concave programming problem

$$(C.P.) \quad \begin{aligned} & \text{maximize } f(y) \\ & g_i(y) \leq \bar{z}_i, \quad i = 1, \dots, m, \\ & y \in Y, \end{aligned}$$

where  $Y$  is a convex set of  $\mathbb{R}^n$ ,  $f$  is a concave function defined on  $Y$ ,  $g_i$ ,  $i = 1, \dots, m$ , are convex functions defined on  $Y$  and  $\bar{z}_i$ ,  $i = 1, \dots, m$ , are given numbers, we say that the *Slater condition* or *Slater constraint qualification* is satisfied for (C.P.) if there exists an element  $\hat{y} \in Y$  such that  $g_i(\hat{y}) < \bar{z}_i$ ,  $i = 1, \dots, m$ .

**Theorem 8.** If the value  $v(z)$  is finite and if problem (C.P.) satisfies the Slater condition, then (C.P.) is stable.

**Proof of Theorem 6.**

**Sufficiency.** Let us consider the trajectory  $\{x^*(t)\}_{t=0}^N \in X_T(x^0, N)$ .

As  $(x(t), x(t+1)) \in T$ ,  $t = 0, 1, \dots, N-1$ , and  $x(0) = x^*(0)$ , from condition 2 we get

$$\begin{aligned} p(1)x^*(0) - p(1)x(1) &\geq 0 \\ p(1)x(1) - p(2)x(2) &\geq 0 \\ &\vdots \\ p(N-1)x(N-1) - p(N)x(N) &\geq 0. \end{aligned}$$

Putting together all these inequalities, we get

$$p(0)x^*(0) \geq p(N)x(N).$$

Taking conditions 1 and 3 into account, it holds

$$px^*(N) \geq px(N),$$

which shows the  $p$ -optimality of the trajectory  $\{x^*(t)\}_{t=0}^N$ .

**Necessity.** First we prove that for every  $t = 0, 1, \dots, N-1$ , we can find a vector  $p(t) \in \mathbb{R}_+^n$  such that

$$\max_{(x,y) \in \tilde{T}} [V_N^t(y) - p(t-1)x] = 0,$$

where the maximum is attained in  $(x, y) = (x^*(t-1), x^*(t))$ .

From this, on the ground of Lemma 9,

$$V_N^t(x^*(t)) = px^*(N), \quad t = 0, 1, \dots, N-1,$$

we shall obtain

$$p(0)x^*(0) = p(1)x^*(1) = \dots = p(N-1)x^*(N-1) = px^*(N).$$

In the same time we prove that from the construction of vector  $p(t)$  we get conditions 2 and 3 of the theorem. Before proving the existence of vector  $p(t)$ ,  $t = 0, 1, \dots, N$ , with the desired properties, we remark that conditions 2 and 3 imply  $p(t) \geq [0]$ ,  $t = 0, 1, \dots, N-1$ .

Let us absurdly suppose that for a certain index  $0 \leq t \leq N-1$ , we have  $p(t) = [0]$ . Let be  $(\hat{x}, \hat{y}) \in T$  a process such that  $\hat{y} > [0]$ . Condition 2 implies  $p(t+1)\hat{y} \leq 0$ , inequality which implies  $p(t+1) = [0]$ . By reiterating this argument, we get finally the contradiction  $p = p(N) = [0]$ .

Let us consider the concave optimization problem

$$\text{maximize } V_N^1(y) \tag{57}$$

$$\text{subject to } (x^*(0), y) \in \tilde{T}. \quad (58)$$

From Lemma 9 the optimal value of this problem  $V_N^0(x^*(0))$ , being  $y = x^*(1)$  the optimal solution. On the ground of this lemma, the perturbation function of problem (57)-(58) (see Definition 17), defined on  $\mathbb{R}_+^n$ , is

$$V_N^0(x) = \max \left\{ V_N^1(y) / (x, y) \in \tilde{T} \right\}.$$

As  $x^*(0) = x^0 > [0]$ , function  $V_N^0$  admits a subgradient at  $x = x^*(0)$ . So, problem (57)-(58) is stable and, on the ground of Theorem 7, the Kuhn-Tucker theorem holds. Then exists therefore a vector  $p(0) \in \mathbb{R}_+^n$  such that

$$\max_{(x,y) \in \tilde{T}} [V_N^1(y) - p(0) (x - x^*(0))] = V_N^0(x^*(0)), \quad (59)$$

with the maximum attained at  $(x, y) = (x^*(0), x^*(1))$ .

Being  $p(0)$  a subgradient of the function  $V_N^0$  at  $x = x^*(0)$ , the following inequality is satisfied:

$$V_N^0(x^*(0)) - p(0) x^*(0) \geq V_N^0(x) - p(0) x, \text{ for any } x \in \mathbb{R}_+^n. \quad (60)$$

The function  $V_N^0(x) - p(0) x$  is positively homogeneous on  $\mathbb{R}_+^n$ , so its maximum can be only or zero or infinite. It results

$$V_N^0(x^*(0)) - p(0) x^*(0) = 0. \quad (61)$$

From this and from (60), taking into account that  $V_N^0(x) \geq 0$ , for each  $x \in \mathbb{R}_+^n$  we obtain  $p(0) \in \mathbb{R}_+^n$ .

From equality (61), relation (59) can be written as

$$\max_{(x,y) \in \tilde{T}} [V_N^0(y) - p(0) x] = 0, \quad (62)$$

with the maximum attained at  $(x, y) = (x^*(0), x^*(1))$ .

Now let us suppose that for a given index  $1 \leq t \leq N - 1$  we have build a vector  $p(t - 1) \in \mathbb{R}_+^n$  for which

$$\max_{(x,y) \in \tilde{T}} [V_N^t(y) - p(t - 1) x] = 0, \quad (63)$$

with the maximum attained at  $(x, y) = (x^*(t), x^*(t + 1))$ .

As  $V_N^t(y) = \max_{(y,u) \in \tilde{T}} V_N^{t+1}(u)$  relation (63) can be written as

$$\max_{\substack{(x,y) \in \tilde{T} \\ (y,u) \in \tilde{T}}} [V_N^{t+1}(u) - p(t - 1) x] = 0, \quad (64)$$

with the maximum attained at  $(x, y) = (x^*(t-1), x^*(t))$ ,  $(x, u) = (x^*(t), x^*(t+1))$ .

Taking into account the definition of the set  $\tilde{T}$ , relation (64) shows that the problem

$$(P_t) \quad \left\{ \begin{array}{l} \text{maximize } [V_N^{t+1}(u) - p(t-1)x], \\ \text{subject to: } z - y \leq [0] \\ \quad \quad \quad u - w \leq [0] \\ \quad \quad \quad (x, y) \in T \\ \quad \quad \quad (z, w) \in T \\ \quad \quad \quad u \in \mathbb{R}_+^n \end{array} \right.$$

has optimal value equal zero, the optimal solutions being  $(x, y) = (x^*(t-1), x^*(t))$ ,  $(z, w) = (x^*(t), x^*(t+1))$ ,  $u = x^*(t+1)$ .

The problem satisfies the solution Slater condition. From assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_4)$ , it results the existence of a process  $(\tilde{x}, \tilde{y}) \in T$  with  $\tilde{x} > [0]$ ,  $\tilde{y} > [0]$ . Choosing a number  $k > 0$  such that  $k\tilde{y} > \tilde{x}$  and taking  $(x, y) = (k\tilde{x}, k\tilde{y})$ ,  $(z, w) = (\tilde{x}, \tilde{y})$ ,  $u = [0]$ , the two inequality constraints of problem  $(P_t)$  will be satisfied with the strict inequality. From Theorem 8 it results that the Kuhn-Tucker theorem holds for this problem. There exist therefore multipliers  $p(t), p'(t) \in \mathbb{R}_+^n$  such that

$$V_N^{t+1}(u) - p(t-1)x - p'(t)(z - y) - p'(t)(u - w) \leq 0 \quad (65)$$

for every  $(x, y) \in T$ ,  $(z, w) \in T$ ,  $u \in \mathbb{R}_+^n$ .

In (65) the equality (to zero) is obtained for  $(x, y) = (x^*(t-1), x^*(t))$ ,  $(z, w) = (x^*(t), x^*(t+1))$ ,  $u = x^*(t+1)$ .

Taking  $(x, y) = ([0], [0])$ ,  $u = w$ , relation (65) becomes

$$V_N^{t+1}(w) - p(t)z \leq 0 \text{ for every } (z, w) \in T. \quad (66)$$

For  $(z, w) = ([0], [0])$ ,  $u = [0]$  relation (65) gives

$$p(t-1)x - p(t)y \geq 0 \text{ for every } (x, y) \in T. \quad (67)$$

Finally, taking in (65)  $(x, y) = (x^*(t-1), x^*(t))$ ,  $(z, w) = (x^*(t), x^*(t+1))$ ,  $u = x^*(t+1)$ , we get

$$V_N^{t+1}(x^*(t+1)) - p(t)x^*(t) - p(t-1)x^*(t-1) + p(t)x^*(t) = 0. \quad (68)$$

Taking (67) and (68) into account, as both quantities between parentheses of the two equalities, are nonpositive, we obtain

$$V_N^{t+1}(x^*(t+1)) - p(t)x^*(t) = 0 \quad (69)$$

$$p(t-1)x^*(t-1) + p(t)x^*(t) = 0. \quad (70)$$

From (66) and (69) it results

$$\max_{(x,y) \in T} [V_N^{t+1}(y) - p(t)x] = 0, \quad (71)$$

the maximum being attained at  $(x, y) = (x^*(t), x^*(t+1))$ .

It remains to prove the existence of a vector  $p(N-1) \in \mathbb{R}_+^n$  with the property

$$p(N-2)x - p(N-1)y \geq 0, \quad p(N-1)x - py \geq 0,$$

for every  $(x, y) \in T$  and

$$p(N-1)x^*(N-1) = px^*(N).$$

We have already proved that

$$\max_{(x,y) \in \tilde{T}} [V_N^{N-1}(y) - p(N-2)x] = 0, \quad (72)$$

the maximum being attained at  $(x, y) = (x^*(N-2), x^*(N-1))$ .

By substituting

$$V_N^{N-1}(y) = \max_{(x,y) \in \tilde{T}} pu,$$

relation (72) becomes

$$\max_{\substack{(x,y) \in \tilde{T} \\ (y,u) \in \tilde{T}}} (pu - p(N-2)x) = 0$$

with maximal values given by  $(x, y) = (x^*(N-2), x^*(N-1))$ ,  $(y, u) = (x^*(N-1), x^*(N))$ .

Recalling the definition of the technological set  $T$ , (72) puts in to evidence that the concave programming problem

$$(P_{N-1}) \quad \left\{ \begin{array}{l} \text{maximize } [pu - p(N-2)x] \\ \text{subject to: } z - y \leq [0] \\ \quad \quad \quad u - w \leq [0] \\ \quad \quad \quad (x, y) \in T \\ \quad \quad \quad (z, w) \in T \\ \quad \quad \quad u \in \mathbb{R}_+^n \end{array} \right.$$

has optimum value equal to zero, the optimal solution being  $(x, y) = (x^*(N-2), x^*(N-1))$ ,  $(z, w) = (x^*(N-1), x^*(N))$ ,  $u = x^*(N)$ .

We know that this problem satisfies the Slater condition. By the Kuhn-Tucker theorem there exist vector  $p(N-1), p'(N-1) \in \mathbb{R}_+^n$  such that

$$pu - p(N-2)x - p(N-1)(z - y) - p'(N-1)(u - w) \leq 0 \quad (73)$$

for every  $(x, y) \in T$ ,  $(z, w) \in T$ ,  $u \in \mathbb{R}_+^n$ , obtaining the equality (to zero) for  $(x, y) = (x^*(N-2), x^*(N-1))$ ,  $(z, w) = (x^*(N-1), x^*(N))$ ,  $u = x^*(N)$ . Taking  $(x, y) = ([0], [0])$ ,  $u = w$ , relation (73) implies

$$p(N-1)z - pw \geq 0, \text{ for every } (z, w) \in T. \quad (74)$$

By substituting  $(z, w) = ([0], [0])$ ,  $u = [0]$ , relation (73) gives

$$p(N-2)x - p(N-1)y \geq 0, \text{ for every } (x, y) \in T. \quad (75)$$

Finally, for  $(x, y) = (x^*(N-2), x^*(N-1))$ ,  $(z, w) = (x^*(N-1), x^*(N))$ ,  $u = x^*(N)$ , relation (73) becomes

$$(px^*(N) - p(N-1)x^*(N-1)) + p(N-1)x^*(N-1) - p(N-2)x^*(N-2) = 0.$$

From this relation and from (74) and (75) we obtain

$$px^*(N) - p(N-1)x^*(N-1) = 0,$$

$$p(N-1)x^*(N-1) - p(N-2)x^*(N-2) = 0,$$

so that  $p(N-1)$  satisfies all the required conditions. □

Given the vectors  $x^0 \in \mathbb{R}_+^n$ ,  $p \in \mathbb{R}_+^n$ , the  $p$ -optimal trajectories of  $X_T(x^0, N)$  are optimal solutions of the problem

$$(P) \quad \begin{cases} \text{maximize } px(N), \\ \text{subject to: } (x(t), x(t+1)) \in T, \\ \quad \quad \quad t = 0, 1, \dots, N-1, \quad x(0) = x^0. \end{cases}$$

We want to link this problem to another optimization problem, whose optimal solutions are vectors  $p(t)$ ,  $t = 0, 1, \dots, N$ , which appear in Theorem 6. We first define the following set

$$T_D = \{(q, p) : (q, p) \in \mathbb{R}_+^n \times \mathbb{R}_+^n, qx - py \geq 0, \forall (x, y) \in T\}, \quad (76)$$

set we may call *dual technological set of T*.

**Lemma 10.** If the technology  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$ ,  $(T_3)$  and  $(T_4)$ , then  $T_D$  verifies assumptions  $(T_1)$ ,  $(T_2)$ ,  $(T_4)$  and  $(T_5)$ . Moreover, for every  $p \in \mathbb{R}_+^n$ , there exists a vector  $q \in \mathbb{R}_+^n$ , such that  $(q, p) \in T_D$ .

**Proof.** From (76) it results at once that  $T_D$  is a closed convex cone which satisfies assumption  $(T_5)$ . In order to prove that  $T_D$  satisfies assumption  $(T_2)$ , let  $([0], p) \in T_D$ . Taking (76) into

account, we find  $-py \geq 0$  for every  $(x, y) \in T$ . As  $T$  satisfies assumption  $(T_4)$ , there exists  $(\hat{x}, \hat{y}) \in T$ , with  $\hat{y} > [0]$ . Therefore  $-p\hat{y} \geq 0$ , which implies  $p = [0]$ .

It remains to prove that for every  $p \in \mathbb{R}_+^n$  there exists  $q \in \mathbb{R}_+^n$  with  $(q, p) \in T_D$ . By choosing  $\bar{x} \in \mathbb{R}_+^n$ ,  $\bar{x} > [0]$ , let us consider the problem **maximize**  $py$ , subject to  $(\bar{x}, y) \in T$ . Let  $(\bar{x}, \bar{y})$  be an optimal solution of this problem. By Theorem 6, there exists a vector  $q \in \mathbb{R}_+^n$  such that  $q\bar{x} - p\bar{y} = 0$  and  $qx - py \geq 0$  for every  $(x, y) \in T$ . Therefore  $(q, p) \in T_D$ .

□

Given the pair  $(q, p) \in T_D$ , we can interpret vector  $q$  as a system of prices appointed to the input goods, whereas  $p$  as a system of prices appointed to the output goods. Therefore  $T_D$  consists of the set of price vectors  $(p, q)$  related to those production processes of  $T$  that yield no gain. Now let us define the problem

$$(D) \quad \begin{cases} \text{minimize } p(0)x^0, \\ \text{subject to: } (p(t), p(t+1)) \in T_D, \\ \quad \quad \quad t = 0, 1, \dots, N-1, p(N) = p, \end{cases}$$

we may call *dual problem of (P)*.

Let  $\{x(t)\}_{t=0}^N$  and  $\{p(t)\}_{t=0}^N$  be, respectively, feasible solutions of problems (P) and (D). We have  $p(0)x^0 \geq p(1)x(1) \geq \dots \geq p(N-1)x(N-1) \geq px(N)$ .

Hence the equality

$$p(0)x^0 = px(N)$$

implies that  $\{x(t)\}_{t=0}^N$  is an optimal solution of (P), whereas  $\{p(t)\}_{t=0}^N$  is an optimal solution of (D). We obtain the following variant version of Theorem 6.

**Theorem 9.** Let  $\{x^*(t)\}_{t=0}^N$  be a feasible solution of problem (P). A sufficient condition for  $\{x^*(t)\}_{t=0}^N$  to be optimal solution of the same problem, is that there exists a feasible solution of problem (D),  $\{p(t)\}_{t=0}^N$ , such that  $p(0)x^0 = px^*(N)$ .

If  $x^0 > [0]$  and if  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$ ,  $(T_3)$  and  $(T_4)$ , this condition is also necessary.

## 6. Efficient Growth

The concept of *efficiency* was first used in an intertemporal context by Debreu (1951) and Malinvaud (1953, 1962). See also Dorfman, Samuelson and Solow (1958), Furuya and Inada (1962), Nikaido (1968). The definition of the concept of efficient growth is based on the notion of *Pareto maximum point* or *Pareto efficient point*.

**Definition 18.** Let be given a set  $\Omega \subset \mathbb{R}^n$ . An element  $x \in \Omega$  is a Pareto maximum point for  $\Omega$  if there is no elements  $x \in \Omega$  such that  $x \geq x^*$ .

Let be given an initial state of a Gale - von Neumann model, say  $x^0 \in \mathbb{R}_+^n$  and let be given the set  $B_T(x^0, N)$  of the states that can be obtained from  $x^0$  after  $N$  periods.

**Definition 19.** A growth trajectory  $\{x^*(t)\}_{t=0}^N \in X_T(x^0, N)$  is called efficient if  $x^*(N)$  is a Pareto maximum point for the set  $B_T(x^0, N)$ .

**Lemma 11.** Under assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_3)$ , for every  $x^0 \in \mathbb{R}_+^n$ , the set  $X_T(x^0, N)$  contains at least an efficient trajectory.

**Proof.** Let  $p \in \mathbb{R}^n$ ,  $p > [0]$ . We prove that every  $p$ -optimal trajectory of  $X_T(x^0, N)$  is efficient. Let  $\{x^*(t)\}_{t=0}^N$  be such a trajectory and let us absurdly suppose that it is not efficient. Therefore there will exist a trajectory  $\{x(t)\}_{t=0}^N \in X_T(x^0, N)$  such that  $x(N) \geq x^*(N)$ .

From this, being  $p > [0]$ , we get  $p x(N) > p x^*(N)$ , which is in contradiction with the  $p$ -optimality of the trajectory  $\{x^*(t)\}_{t=0}^N$ . □

**Remark 9.** Let  $u$  be an increasing preference function:  $x^1 \geq x^2 \Rightarrow u(x^1) > u(x^2)$ . What proved in Lemma 11 can be extended to say that every final optimal trajectory  $\{x^*(t)\}_{t=0}^N$ , with reference to  $u$ , in the set  $X_T(x^0, N)$ , is also efficient in  $X_T(x^0, N)$ .

Now we prove a result which is, in a sense, the converse of Lemma 11.

**Lemma 12.** Let assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_3)$  be satisfied.

- a) If  $\{x^*(t)\}_{t=0}^N$  is an efficient trajectory of  $X_T(x^0, N)$ , then there exists a price vector  $p \geq [0]$  such that  $\{x^*(t)\}_{t=0}^N$  is  $p$ -optimal in  $X_T(x^0, N)$ .
- b) Given a price vector  $p \geq [0]$  among the  $p$ -optimal trajectories of  $X_T(x^0, N)$ , there exists an efficient trajectory.

**Proof.**

- a) Let  $\{x^*(t)\}_{t=0}^N \in X_T(x^0, N)$  an efficient trajectory. The set

$$M = \left\{ x(N) - x^*(N) : \{x(t)\}_{t=0}^N \in X_T(x^0, N) \right\}$$

is convex. As the trajectory  $\{x^*(t)\}_{t=0}^N$  has been supposed efficient, it holds  $M \cap \mathbb{R}_+^n = \{[0]\}$ . Therefore there exists a vector  $p \in \mathbb{R}_+^n$ ,  $p \neq [0]$ , such that

$$pz \leq 0, \quad \forall z \in M, \tag{77}$$

$$pz \geq 0, \quad \forall z \in \mathbb{R}_+^n. \tag{78}$$

From (78) we have  $p \geq [0]$ , whereas (77) shows the  $p$ -optimality of the trajectory  $\{x^*(t)\}_{t=0}^N$ .

b) Let us denote by  $Q$  the set of  $p$ -optimal trajectories of  $X_T(x^0, N)$ . Let be

$$\theta = \max \left\{ px(N) : \{x(t)\}_{t=0}^N \in X_T(x^0, N) \right\}.$$

Therefore

$$Q = \left\{ \{x(t)\}_{t=0}^N : \{x(t)\}_{t=0}^N \in X_T(x^0, N), px(N) = \theta \right\}.$$

It is easy to see that the set  $Q$  is compact. Let us consider a vector  $q \in \mathbb{R}_+^n$ ,  $q > [0]$ . Let  $\{x^*(t)\}_{t=0}^N$  be an optimal solution of the problem

$$\text{maximize } qx(N) \tag{79}$$

$$\text{subject to } \{x(t)\}_{t=0}^N \in Q. \tag{80}$$

We prove that the trajectory  $\{x^*(t)\}_{t=0}^N$  is efficient. Let us absurdly suppose that the said trajectory is not efficient. It will exist a trajectory  $\{x(t)\}_{t=0}^N \in X_T(x^0, N)$ , such that  $x(N) \geq x^*(N)$ . From this we deduce

$$px(N) \geq px^*(N), \tag{81}$$

$$qx(N) \geq qx^*(N). \tag{82}$$

From (81) we get  $\{x(t)\}_{t=0}^N \in Q$ . This relation, together with (82), contradicts the assumption that the trajectory  $\{x^*(t)\}_{t=0}^N$  is an optimal solution of the problem (79)-(80). □

Lemma 12, together with Theorem 6, allows to formulate the following conditions of efficiency.

**Theorem 10.** Let us assume that the technology  $T$  verifies assumptions  $(T_1)$ ,  $(T_2)$ ,  $(T_3)$  and  $(T_4)$ . Let be  $x^0 > [0]$ . In order that a trajectory  $\{x^*(t)\}_{t=0}^N \in X_T(x^0, N)$ , is efficient, it is necessary that there exist vectors  $p(t) \geq [0]$ ,  $t = 0, 1, \dots, N$ , such that:

- a)  $p(0)x^*(0) = p(1)x^*(1) = \dots = p(N)x^*(N)$ ,
- b)  $p(t)x - p(t+1)y \geq 0$  for every  $(x, y) \in T$ ,  $t = 0, 1, \dots, N$ .

If  $p(N) > [0]$ , then conditions a) and b) are also sufficient.

**Definition 20.** Let be given the efficient trajectory  $\{x^*(t)\}_{t=0}^N \in X_T(x^0, N)$ ; every set of vectors  $p(t) \in \mathbb{R}^n$ ,  $p(t) \geq [0]$ ,  $t = 0, 1, \dots, N$ , which satisfies conditions a) and b) of Theorem 10, is called sequence of efficient prices associated to the said trajectory.

Now we shall define a concept of efficiency for growth trajectories of infinite horizon.

**Definition 21.** Let be given a growth trajectory of infinite horizon  $\{x(t)\}_{t=0}^\infty \in X_T(x^0)$ . For every index  $N \geq 1$ , the trajectory  $\{x(t)\}_{t=0}^N \in X_T(x^0, N)$  is called  $N$ -opening of the trajectory  $\{x(t)\}_{t=0}^\infty$ .

**Definition 22.** A growth trajectory of infinite horizon  $\{x^*(t)\}_{t=0}^\infty \in X_T(x^0)$  is called *intertemporally efficient* if every its opening is efficient.

**Remark 10.** Let  $\{x(t)\}_{t=0}^\infty$ ,  $x(t) = \alpha^t(T)x$ ,  $t = 0, 1, \dots$ , a maximal growth trajectory sustained by a von Neumann price vector  $p > [0]$ . On the ground of Theorem 10 it is easy to see that the trajectory  $\{x(t)\}_{t=0}^\infty$  is intertemporally efficient in  $X_T(x)$ . Before discussing the problem of the existence of a trajectory intertemporally efficient, we shall prove that under the assumptions of the Theorem 9, to every trajectory intertemporally efficient we can associate a sequence of efficient prices.

**Theorem 11.** Let us suppose that the technology  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$ ,  $(T_3)$  and  $(T_4)$ . Let  $x^0 > [0]$ . If the trajectory  $\{x^*(t)\}_{t=0}^\infty \in X_T(x^0)$  is intertemporally efficient then there exists a sequence of prices  $\{p(t)\}_{t=0}^\infty \in \mathbb{R}_+^n$ ,  $p(t) \geq [0]$ ,  $t = 0, 1, \dots$ , such that:

- a)  $p(0)x^*(0) = \dots = p(t-1)x^*(t-1) = p(t)x^*(t) = \dots$ ,
- b)  $p(t)x - p(t+1)y \geq 0$ , for every  $(x, y) \in T$ ,  $t = 0, 1, \dots$ .

**Proof.** Being the trajectory  $\{x^*(t)\}_{t=0}^\infty$  intertemporally efficient, every its opening is efficient. Being satisfied all assumptions of Theorem 10, to each opening there is associated a sequence of efficient prices. Let be  $N \geq 1$ . For each index  $M \geq 1$  we denote by  $P_{N+M}$  the set of sequences of efficient prices associated to the opening  $\{x^*(t)\}_{t=0}^{N+M}$ . Therefore  $P_{N+M}$  consists of all sequences of semipositive vectors  $(p(0), \dots, p(N), \dots, p(N+M))$  such that

$$p(0)x^*(0) = \dots = p(N)x^*(N) = \dots = p(N+M)x^*(N+M) \quad (83)$$

and

$$p(t)x - p(t+1)y \geq 0, \text{ for every } (x, y) \in T, t = 0, 1, \dots, N+M-1. \quad (84)$$

To the sets  $P_{N+M}$  we associate the set  $Q_{N,M}$ , a sequence of vectors  $(p(0), \dots, p(N))$  with the properties:

- 1) 
$$\sum_{t=0}^N \sum_{i=1}^n p_i(t) = 1.$$
- 2) For a given number  $\mu > 0$  and for a given sequence  $(p(N+1), \dots, p(N+M))$ , the sequence  $(\mu p(0), \dots, \mu p(N), p(N+1), \dots, p(N+M))$  is in  $P_{N+M}$ .

The sets  $Q_{N,M}$  are nonempty, convex and compact. Moreover,  $Q_{N,M+1} \subseteq Q_{N,M}$ . It results that the set

$$Q_{N,\infty} = \bigcap_{M=1}^{\infty} Q_{N,M} \quad (85)$$

is nonempty.

Let  $(p(0), \dots, p(N)) \in Q_{N, \infty}$ . We prove that there exists a number  $\lambda > 0$  and a vector  $p'(N+1)$  such that  $(\lambda p(0), \dots, \lambda p(N), p'(N+1)) \in Q_{N+1, \infty}$ . As  $(p(0), \dots, p(N)) \in \bigcap_{M=1}^{\infty} Q_{N, M}$ , for every  $M$  there exist vectors  $p^M(N+1), \dots, p^M(N+M)$  and numbers  $\mu_M > 0$  with the property  $(\mu_M p(0), \dots, \mu_M p(N), p^M(N+1), \dots, p^M(N+M)) \in P_{N+M}$ .

From this, taking in to account that for  $M \geq 2$ , it holds  $P_{N+M} = P_{N+1+(M-1)}$ , we obtain

$$(\alpha_M \mu_M p(0), \dots, \alpha_M \mu_M p(N), \alpha_M p^M(N+1)) \in Q_{N+1, M-1}, \quad (86)$$

where

$$\alpha_M = \frac{1}{\mu_M + \sum_{i=1}^n p_i^M(N+\alpha)}. \quad (87)$$

The equality (87) implies the boundedness of the sequences  $\{\alpha_M \mu_M\}_{M=2}^{\infty}$ ,  $\{\alpha_M p^M(N+1)\}_{M=2}^{\infty}$ . Let  $\alpha$  and  $p'(N+1)$  be, respectively, limit points of the said sequences. From (86) and being  $Q_{N+1, M-1}$  compact and included one in the other, we obtain

$$(\alpha p(0), \dots, \alpha (p(N), p'(N+1))) \in \bigcap_{M=1}^{\infty} Q_{N+1, M-1} = Q_{N+1, \infty}.$$

Finally, by means of the sets  $Q_{M, \infty}$ ,  $M = 1, 2, \dots$ , we build a sequence  $\{p(t)\}_{t=0}^{\infty}$  with the requested properties. Let us choose  $((p(0), p(1)) \in Q_{1, \infty}$ . On the basis of what we have seen, there exist a number  $\lambda_1 > 0$  and a vector  $p'(2)$  such that

$$(\lambda_1 p(0), \lambda_1 (p(1), p'(2))) \in Q_{2, \infty}. \quad (88)$$

Let us denote  $p(2) = p'(2)/\lambda_1$ . From the definition of  $Q_{2, \infty}$  it is easy to see that

$$p(0) x^*(0) = p(1) x^*(1) = p(2) x^*(2)$$

and

$$p(t) x - p(t+1) y \geq 0, \text{ for every } (x, y) \in T, t = 0, 1.$$

Relation (88) implies the existence of a number  $\lambda_2 > 0$  and a vector  $p'(3)$  such that

$$(\lambda_2 \lambda_1 p(0), \lambda_2 \lambda_1 p(1), \lambda_2 p'(2), p'(3)) \in Q_{3, \infty}. \quad (89)$$

Let us denote  $p(3) = p'(3)/\lambda_2 \lambda_1$ . Relation (89), together with the construction of vectors  $p(2)$  and  $p(3)$ , gives

$$p(0) x^*(0) = p(1) x^*(1) = p(2) x^*(2) = p(3) x^*(3)$$

and

$$p(t) x - p(t+1) y \geq 0, \text{ for every } (x, y) \in T, t = 0, 1, 2.$$

By iterating this procedure we obtain a sequence of vectors  $\{p(t)\}_{t=0}^{\infty}$  with the desired properties. □

Let us consider a trajectory  $\{x(t)\}_{t=0}^{\infty} \in X_T(x^0)$  and let  $p \in \mathbb{R}_+^n$  be a von Neumann price vector. As  $(x(t), x(t+1)) \in T$ , it holds  $\alpha(T)px(t) - px(t+1) \geq 0$ . Therefore the sequence of nonnegative numbers  $\{px(t)/\alpha^t(T)\}_{t=0}^{\infty}$  is non-increasing. Therefore the quantity

$$\lim_{t \rightarrow \infty} \frac{px(t)}{\alpha^t(T)}$$

exists and is nonnegative.

**Theorem 12.** Let us suppose that the technology  $T$  satisfies the assumptions  $(T_1)$ ,  $(T_2)$ ,  $(T_3)$  and  $(T_4)$ , that it possesses a von Neumann ray  $\{\lambda \bar{x}\}_{\lambda \geq 0}$  and a von Neumann price vector  $\bar{p}$  such that  $\bar{x} > [0]$ ,  $\bar{p} > [0]$ . Let  $x^0 > [0]$ . Then, for every intertemporally efficient trajectory  $\{x(t)\}_{t=0}^{\infty} \in X_T(x^0)$ , it holds

$$\lim_{t \rightarrow \infty} \frac{\bar{p}x(t)}{\alpha^t(T)} > 0.$$

**Proof.** The assumptions of Theorem 11 being satisfied, there exists a sequence of prices  $\{p(t)\}_{t=0}^{\infty} \subset \mathbb{R}_+^n$ ,  $p(t) \geq [0]$ ,  $t = 0, 1, \dots$ , such that

$$p(0)x(0) = \dots = p(t-1)x(t-1) = p(t)x(t) = \dots \quad (90)$$

and

$$p(t)x(t) - p(t+1)y \geq 0, \text{ for every } (x, y) \in T, t = 0, 1, \dots \quad (91)$$

Without loss of generality, we can suppose  $p(0) \leq \bar{p}$ . Let us consider  $T_D$ , the dual technology of  $T$ , defined as follows:

$$T_D = \{(q, p) : (q, p) \in \mathbb{R}_+^n \times \mathbb{R}_+^n, qx - py \geq 0, \text{ for every } (x, y) \in T\}.$$

For every  $q \in \mathbb{R}_+^n$ , let us consider the sequence of sets  $B_T(q, t)$ ,  $t = 0, 1, 2, \dots$ , defined as follows:

$$B_{T_D}(q, 1) = \{p : (q, p) \in T_D\},$$

$$B_{T_D}(q, t+1) = \{p : (f, p) \in T_D, \text{ for a given } f \in B_{T_D}(q, t)\}.$$

$B_{T_D}(q, t)$  represents the set of the states which, using technology  $T_D$ , allow to reach  $q$ , after  $t$  periods.

We prove that the set

$$A = \bigcup_{t=1}^{\infty} \alpha^t(T)B_{T_D}(\bar{p}, t)$$

is bounded.

Let  $g \in A$ . There exists therefore an index  $t$  for which  $g \in \alpha^t(T)B_{T_D}(\bar{p}, t)$ . By the definition of the set  $B_{T_D}(\bar{p}, t)$ , we deduce the existence of nonnegative vectors  $q(2), q(3), \dots, q(t-1)$  so that

$$(\bar{p}, q(1)) \in T_D, (q(1), q(2)) \in T_D, \dots, \left(q(t-1), \frac{g}{\alpha^t(T)}\right) \in T_D. \quad (92)$$

On the other hand

$$(\bar{x}, \alpha(T)\bar{x}) \in T, (\alpha(T)\bar{x}, \alpha^2(T)\bar{x}) \in T, \dots, (\alpha^{t-1}(T)\bar{x}, \alpha^t(T)\bar{x}) \in T. \quad (93)$$

Relations (92), (93) and the definition of the technology  $T_D$ , imply

$$\bar{p}\bar{x} \geq \alpha(T)q(1)\bar{x} \geq \alpha^2(T)q(2)\bar{x} \geq \dots \geq \alpha^{t-1}(T)q(t-1)\bar{x} \geq \alpha^t(T)\frac{g\bar{x}}{\alpha^t(T)}.$$

Therefore

$$\bar{p}\bar{x} \geq g\bar{x}.$$

As  $\bar{x} > [0]$ , we have  $\min\{h\bar{x}, \|h\| = 1\} = n > 0$ . We obtain

$$\|g\| \leq \frac{1}{n} \frac{g\bar{x}}{\|g\|} \|g\| = \frac{1}{n} g\bar{x} \leq \frac{1}{n} \bar{p}\bar{x},$$

so that the set  $A$  is bounded. From the assumption  $\bar{p} > [0]$  it results the existence of a number  $\lambda > 0$  with the property  $g \leq \lambda\bar{p}$  for every  $g \in A$ . Relation (91) shows that  $p(t) \in B_{T_D}(p(0), t)$ ,  $t = 1, 2, \dots$ . By Lemma 10, technology  $T_D$  satisfies assumption (T<sub>5</sub>). The inequality  $p(0) \leq \bar{p}$  implies the inclusion  $B_{T_D}(p(0), t) \subset B_{T_D}(\bar{p}, t)$ ,  $t = 1, 2, \dots$ . It results

$$p(t) \in B_{T_D}(\bar{p}, t), \quad t = 1, 2, \dots,$$

so that

$$\alpha^t(T)p(t) \in A, \quad t = 1, 2, \dots$$

From this we obtain

$$\alpha^t(T)p(t) \leq \lambda\bar{p}, \quad t = 1, 2, \dots$$

Therefore

$$\frac{\bar{p}x(t)}{\alpha^t(T)} \geq p(t)x(t) = p(0)x(0) > 0, \quad t = 1, 2, \dots,$$

and finally we obtain

$$\lim_{t \rightarrow \infty} \frac{\bar{p}x(t)}{\alpha^t(T)} > 0.$$

□

The assumptions made since now on technology  $T$  are not sufficient to assure the existence of an intertemporally efficient trajectory. They are not even sufficient for the opening of an efficient trajectory to be efficient.

A further assumption which assures the efficiency of each opening of an efficient trajectory is *primitivity* ( $\mathbf{T}_7$ ). We recall here this assumption with the notations since here introduced.

**Assumption ( $\mathbf{T}_7$ ).** Technology  $T$  is *primitive*, i.e. for each  $x^0 \geq [0]$  there exist an index  $s \geq 1$  and a trajectory  $\{x(t)\}_{t=0}^s \in X_T(x, s)$  such that  $x(s) > [0]$ .

We note that a primitive trajectory satisfies assumption ( $\mathbf{T}_3$ ). Moreover, a primitive technology has the property that, given  $x \geq [0]$ , there exists  $y \geq [0]$  for which  $(x, y) \in T$ .

**Lemma 13.** Let us consider technology  $T$ , where assumptions ( $\mathbf{T}_1$ ), ( $\mathbf{T}_2$ ) and ( $\mathbf{T}_7$ ) hold. Let  $\{x^*(t)\}_{t=0}^N$ ,  $N > 1$ , an efficient technology of  $X_T(x^0, N)$ . Then, for each  $1 \leq M < N$ , the trajectory  $\{x^*(t)\}_{t=0}^M$  is efficient in  $X_T(x^0, M)$ .

**Proof.** Let us absurdly suppose that  $\{x^*(t)\}_{t=0}^M$  is not efficient in  $X_T(x^0, M)$ . Therefore there exists a trajectory  $\{x(t)\}_{t=0}^M \in X_T(x^0, M)$  such that  $x(M) > x^*(M)$ . Let  $y(M) = x(M) - x^*(M)$ . As  $y(M) \geq [0]$ , from assumption ( $\mathbf{T}_7$ ) it results the existence of semipositive vectors  $y(M+1), \dots, y(N)$  such that

$$(y(t), y(t+1)) \in T, \quad t = M, M+1, \dots, N-1.$$

Let us consider the sequence  $\{\bar{x}(t)\}_{t=0}^N$  defined as follows

$$\bar{x}(t) = \begin{cases} x(t), & \text{for } 0 \leq t \leq M, \\ x^*(t) + y(t), & \text{for } M < t \leq N. \end{cases}$$

We obtain  $\{\bar{x}(t)\}_{t=0}^N \in X_T(x^0, N)$ ,  $\bar{x}(N) > x^*(N)$ , in contradiction with the assumed efficiency in  $X_T(x^0, N)$  of the trajectory  $\{x^*(t)\}_{t=0}^N$ . □

In order to assure the existence of intertemporally efficient trajectories, we shall use as variant of assumption ( $\mathbf{T}_7$ ), i.e. the following more restrictive assumption.

**Assumption ( $\mathbf{T}_7$ )'.** For every  $x \geq [0]$  there exists a vector  $y > [0]$  such that  $(x, y) \in T$ .

**Theorem 13.** If technology  $T$  satisfies assumptions ( $\mathbf{T}_1$ ), ( $\mathbf{T}_2$ ) and ( $\mathbf{T}_7$ )', then for any  $x^0 \in \mathbb{R}_+^n$ ,  $X_T(x^0)$  contains an intertemporally efficient trajectory.

**Proof.** We denote by  $S$  the space of sequences of elements of  $\mathbb{R}_+^n$  equipped with the metric

$$\rho(x, y) = \sum_{t=0}^{\infty} \frac{\|x(t) - y(t)\|}{1 + \|x(t) - y(t)\|} \cdot \frac{1}{2^{t+1}},$$

where  $x = \{x(t)\}_{t=0}^{\infty}$ ,  $y = \{y(t)\}_{t=0}^{\infty}$ . For each set  $Q \subset S$  and each index  $\tau = 0, 1, \dots$ , we denote

$$Q_{\tau} = \{x(\tau) : \{x(t)\}_{t=0}^{\infty} \in Q\}.$$

The set  $Q$  is compact in  $S$  if and only if is closed and every set  $Q_{\tau}$ ,  $\tau = 0, 1, \dots$ , is bounded in  $\mathbb{R}^n$ . For  $N = 1, 2, \dots$ , we define the following subsets of the set  $X_T(x^0)$ :

$$Y(N) = \left\{ \{x(t)\}_{t=0}^{\infty} : \{x(t)\}_{t=0}^{\infty} \in X_T(x^0), \{x(t)\}_{t=0}^N \text{ is efficient in } X_T(x^0, N) \right\}.$$

On the ground of Lemma 11 these sets are nonempty, whereas by Lemma 13 we have the inclusion

$$Y(N) \subset Y(N-1), \quad N = 1, 2, \dots.$$

On the other hand, from Lemma 5 it results that the sets  $Y_{\tau}(N)$ ,  $\tau = 0, 1, \dots$ , are bounded. Let  $\bar{Y}(N)$  denote the closure in  $S$  of  $Y(N)$ . We now prove the inclusion

$$\bar{Y}(N) \subset Y(N-1). \tag{94}$$

Let  $\{x(t)\}_{t=0}^{\infty} \in \bar{Y}(N)$ . Therefore there exists a sequence  $\{x^n(t)\}_{t=0}^{\infty} \in Y(N)$ ,  $n = 1, 2, \dots$ , such that

$$\lim_{t \rightarrow \infty} x^n(t) = x(t), \quad t = 1, 2, \dots.$$

Let us absurdly suppose that  $\{x(t)\}_{t=0}^{\infty} \notin Y(N-1)$ . There exists therefore a trajectory  $\{\bar{x}(t)\}_{t=0}^{N-1}$  such that  $x(N-1) > \bar{x}(N-1)$ . By assumption  $(T_7)$  it results the existence of an element  $\bar{x}(N)$ , with  $(\bar{x}(N-1), \bar{x}(N)) \in T$  and

$$\bar{x}(N) > x(N). \tag{95}$$

As

$$\lim_{n \rightarrow \infty} x^n(N) = x(N),$$

inequality (95) shows that for  $n$  sufficiently large we get  $\bar{x}(N) > x^n(N)$ , which, together with the relation  $\{\bar{x}(t)\}_{t=0}^N \in X_T(x^0, N)$ , contradicts the assumption  $\{x^n(t)\}_{t=0}^{\infty} \in Y(N)$ . Therefore  $\{x(t)\}_{t=0}^{\infty} \in Y(N-1)$  and (94) is satisfied. The sets  $\bar{Y}(N)$  are compact and included, the one into the other. Their intersection will be therefore nonempty. Relation (94) says that every element of the said intersection is an intertemporally efficient trajectory of  $X_T(x^0)$ . □

## 7. Turnpike Theorems

The class of results named ‘‘Turnpike Theorems’’ puts into evidence a remarkable property of an optimal growth trajectory in relation to final states: under certain conditions all best

growth trajectories must be “close” to the von Neumann ray of balanced growth, except possibly for a finite number of periods, number independent of the length of the trajectory. Since the von Neumann ray is characterized as a trajectory maximizing the speed of growth of the economy, the above result may be nicknamed, following Dorfman, Samuelson and Solow (1958), “Turnpike Theorem”, the “turnpike” in this case being the von Neumann ray. A historical description of the studies of this subject may be given as follows. Dorfman, Samuelson and Solow (1958) first presented three models of capital accumulation formulated in terms of linear programming, difference equations and calculus of variations respectively. Except for the linear programming model, the authors linearized the systems by expanding them in Taylor’s series. Unfortunately they made some careless but fatal errors; for their difference equation model a correct proof of the Turnpike Theorem has been given by McKenzie (1963b). After the publication of the book by Dorfman, Samuelson and Solow, several Turnpike Theorems have been established by various authors, e.g. for a Leontief-von Neumann model, by Morishima (1961) and McKenzie (1963a), for Gale-von Neumann model, by Radner (1961), Morishima (1964), Inada (1964), Nikaido (1964), Winter (1967) and others.

For good surveys of the various Turnpike Theorems presented in the literature, see McKenzie (1976, 1986), McKenzie and Yano (1980), Turnovsky (1970), Wan (1971).

For multisectoral optimal growth with consumptions, and related Turnpike Theory, one may see, e.g., the papers of Rader (1975, 1976) and Tsukui (1967).

We shall be concerned only with trajectories of finite length and with two types of Turnpike Theorems:

- 1) The *weak Turnpike Theorem*, or *Radner’s Turnpike Theorem*, which proves that, for any thin neighbouring cone of the von Neumann ray, any optimal trajectory starting at a common initial position, stay within the cone except for a finite number of periods, but without precisising the position of these periods of exception.
- 2) The *strong Turnpike Theorem* or *Nikaido’s Turnpike Theorem* which assures the turnpike property by precisising also the number of periods of exception, initial and final. The strong Turnpike Theorem has been independently proved also by Inada (1964).

We first assume that technology  $T$  satisfies assumptions  $(T_1)$  and  $(T_2)$ . Moreover, we assume that the following set of conditions is satisfied:

**Assumptions (I).** There exist a vector  $\hat{x} \geq [0]$ , a price vector  $\hat{p} \geq [0]$  and a number  $\alpha > 0$  such that:

- a)  $(\hat{x}, \alpha\hat{x}) \in T, \hat{p}\hat{x} > 0$ ;
- b)  $\hat{p}y - \alpha\hat{p}\hat{x} \leq 0$ , for any  $(x, y) \in T$ ;
- c)  $\hat{p}y - \alpha\hat{p}\hat{x} < 0$ , for all  $(x, y) \in T$  that are not proportional to  $(\hat{x}, \alpha\hat{x})$ .

It is easy to see that from Assumptions (I) it results  $\alpha = \alpha(T)$ ,  $\{\lambda\hat{x}\}_{\lambda \geq 0}$  is the unique von Neumann ray of the technology and  $\hat{p}$  is a von Neumann price vector.

On the ground of Theorem 5, the set of Assumptions (I) is satisfied if technology  $T$  satisfies, besides (T<sub>1</sub>) and (T<sub>2</sub>), also (T<sub>3</sub>) and (T<sub>8</sub>), i.e. the technology is strictly convex.

Let  $N$  be the length of the planning period; we shall study the behavior of the  $p$ -optimal trajectories. As already previously seen, this class of trajectories includes the efficient trajectories. Following Radner (1961), we define an “angular distance” between two vectors  $x, y \in \mathbb{R}^n$ ,  $x \neq [0]$ ,  $y \neq [0]$ , as follows:

$$d(x, y) = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|,$$

i.e. the angular distance between two nonzero vectors is the euclidean distance between the two normalized vectors. Without loss of generality, we consider  $\|\hat{x}\| = 1$ , so that the angular distance between a vector  $x \neq [0]$  and the von Neumann ray is

$$\left\| \frac{x}{\|x\|} - \hat{x} \right\|.$$

In order to prove Radner’s Turnpike Theorem we need the following lemma, due to Radner.

**Lemma 14.** Assume that technology  $T$  satisfies assumptions (T<sub>1</sub>) and (T<sub>2</sub>). If also Assumptions (I) are satisfied, then for every  $\varepsilon > 0$  there exists  $\delta_\varepsilon \in (0, 1)$  such that  $\hat{p}y \leq (1 - \delta_\varepsilon)\hat{p}x$ , for every  $(x, y) \in T$  with  $\left\| \frac{x}{\|x\|} - \hat{x} \right\| \geq \varepsilon$ .

**Proof.** As  $T$  is a cone, it is sufficient to prove the lemma for processes  $(x, y) \in T$  with  $\|x\| = 1$ . Consider the set

$$V_\varepsilon = \left\{ (x, y) : (x, y) \in T, \|x\| = 1, \left\| \frac{x}{\|x\|} - \hat{x} \right\| \geq \varepsilon \right\}.$$

We must show that if this set is nonempty, then there exists a number  $\delta_\varepsilon \in (0, 1)$  with the properties described in the lemma (if  $V_\varepsilon$  is empty, any  $\delta_\varepsilon \in (0, 1)$  will do). From Lemma 5 it results the compactness of set  $V_\varepsilon$ . We note that for  $(x, y) \in V_\varepsilon$ ,  $x$  is not proportional to  $\hat{x}$ . Assumptions (I) imply

$$\hat{p}y < \alpha\hat{p}x, \text{ for every } (x, y) \in V_\varepsilon. \quad (96)$$

Therefore  $\hat{p}x > 0$  for  $(x, y) \in V_\varepsilon$ , so that the function  $f(x, y) = py/px$  is a well-defined continuous function on  $V_\varepsilon$ .

Let  $\tau_\varepsilon$  be the maximum of this function on the set  $V_\varepsilon$ . From (96) it results  $\tau_\varepsilon < \alpha$ . By choosing  $0 < \delta_\varepsilon < \frac{\tau_\varepsilon}{\alpha}$ , we obtain the desired result of the lemma. □

**Definition 23.** Let be given  $x^0 \in \mathbb{R}_+^n$  and the trajectory  $\{\bar{x}(t)\}_{t=0}^\infty \in X_T(x^0)$ . We say that this trajectory *grows at the average rate*  $\alpha - 1$  if

$$\lim_{t \rightarrow \infty} \frac{\widehat{p}\bar{x}(t)}{\alpha^t} > 0.$$

**Remark 11.** Let us suppose that starting from the point  $x^0$  we can reach the turnpike in a given number of periods. In other words, there exists an integer  $N_0 > 0$  and a trajectory  $\{\tilde{x}(t)\}_{t=0}^\infty \in X_T(x^0, N)$  such that  $\tilde{x}(N_0) = \lambda\widehat{x}$ ,  $\lambda > 0$ . Then the trajectory  $\{\bar{x}(t)\}_{t=0}^\infty \in X_T(x^0)$ , defined as follows

$$\bar{x}(t) = \begin{cases} \tilde{x}(t), & \text{for } 0 \leq t \leq N_0, \\ \lambda\alpha^{t-N_0}\widehat{x}, & \text{for } t > N_0, \end{cases}$$

grows with average rate  $\alpha - 1$ . Indeed, it is easy to verify that

$$\lim_{t \rightarrow \infty} \frac{\widehat{p}\bar{x}(t)}{\alpha^t} = \lim_{t \rightarrow \infty} \frac{\lambda\alpha^{t-N_0}\widehat{p}\widehat{x}}{\alpha^t} = \frac{\lambda\widehat{p}\widehat{x}}{\alpha^{N_0}} > 0.$$

Now we prove Radner's Turnpike Theorem or weak Turnpike Theorem.

**Theorem 14.** Consider a technology  $T$  where  $(T_1)$  and  $(T_2)$  are satisfied. Let  $x^0 \in \mathbb{R}_+^n$ ,  $p \in \mathbb{R}_+^n$ . Under Assumptions (I), if:

- a) there exists a trajectory  $\{\bar{x}(t)\}_{t=0}^\infty \in X_T(x^0)$  which grows at an average rate  $\alpha - 1$ ;
- b) there exist numbers  $k' \geq k'' > 0$  such that  $k''\widehat{p} \leq p \leq k'\widehat{p}$ ,

then, for every  $\varepsilon > 0$  there exists a positive integer  $k_\varepsilon$  such that for every  $N$  and every trajectory  $\{x(t)\}_{t=0}^N \in X_T(x^0, N)$ ,  $p$ -optimal, the number of periods in which

$$\left\| \frac{x(t)}{\|x(t)\|} - \widehat{x} \right\| \geq \varepsilon$$

cannot exceed  $k_\varepsilon$ . (The number  $k_\varepsilon$  is independent of the length of the planning period  $N$ ).

**Proof.** Let  $\varepsilon > 0$ . For every  $t$  for which

$$\left\| \frac{x(t)}{\|x(t)\|} - \widehat{x} \right\| \geq \varepsilon$$

from Lemma 14 it results

$$\widehat{p}x(t+1) \leq (1 - \delta_\varepsilon)\alpha\widehat{p}x(t). \tag{97}$$

On the other hand, for every  $t$  from Assumptions (I) we have that the following inequality holds:

$$\widehat{p}x(t+1) \leq \alpha\widehat{p}x(t). \tag{98}$$

Let us suppose that

$$\left\| \frac{x(t)}{\|x(t)\|} - \widehat{x} \right\| \geq \varepsilon$$

for a number  $S$  of periods. From (97) and (98) we obtain

$$\widehat{p}x(N) \leq (1 - \delta_\varepsilon)^S \alpha^N \widehat{p}x^0. \quad (99)$$

We have seen that the sequence  $\left\{ \frac{\widehat{p}\bar{x}(t)}{\alpha^t} \right\}_{t=0}^\infty$  is nonincreasing.

Condition a) implies

$$\lim_{t \rightarrow \infty} \frac{\widehat{p}\bar{x}(t)}{\alpha^t} = \zeta > 0. \quad (100)$$

By condition b), the  $p$ -optimality of the trajectory  $\{x(t)\}_{t=0}^N$  and relation (100), we obtain

$$\frac{\widehat{p}x(N)}{\alpha^N} \geq \frac{px(N)}{k'\alpha^N} \geq \frac{p\bar{x}(N)}{k'\alpha^N} \geq \frac{k''\widehat{p}x(N)}{k'\alpha^N} \geq \frac{k''}{k'}\zeta. \quad (101)$$

Putting together relations (99) and (101) we get

$$\frac{k''}{k'}\zeta \leq (1 - \delta_\varepsilon)^S \widehat{p}x^0.$$

From this we obtain

$$S \leq \frac{\ln\left(\frac{k''\zeta}{k'\widehat{p}x^0}\right)}{\ln(1 - \delta_\varepsilon)}.$$

We remark that condition a) implies  $\widehat{p}x^0 > 0$ . The proof ends, by choosing  $k_\varepsilon$  equal to the integer part of the number  $\ln(k''\zeta/k'\widehat{p}x^0) \cdot [\ln(1 - \delta_\varepsilon)]^{-1}$ . We remark that the number  $k_\varepsilon$  is independent of the length  $N$  of the planning period. □

We note that condition a) is verified, as already remarked, if starting from  $x^0$  it is possible to reach the turnpike in a given number of periods. This surely occurs, for example, when technology  $T$  satisfies assumption  $(T_5)$  and if  $x^0 > [0]$ . Indeed, in this case there exists a number  $\sigma > 0$  such that  $\sigma\widehat{x} \leq x^0$ . As  $(\sigma\widehat{x}, \alpha\sigma\widehat{x}) \in T$ , assumption  $(T_5)$  implies  $(x^0, \alpha\sigma\widehat{x}) \in T$ . It is easy to verify that condition b) is verified if and only if  $\text{supp}(p) = \text{supp}(\widehat{p})$ , where  $\text{supp}(x)$  is the *support* of the semipositive vector  $x$ , i.e.

$$\text{supp}(x) = \{i : x_i > 0\}.$$

Radner's theorem has been criticized because of one dubious point left untouched: an optimal path may several times enter and leave the neighbouring cone at intermediate periods which are far from both the initial and the terminal period.

This kind of behavior is excluded under additional assumptions, in order to state the *strong Turnpike Theorem* or *Nikaido's Turnpike Theorem*. (Nikaido (1964, 1968)).

We assume that technology  $T$  satisfies assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_5)$ . Assumptions (I) are completed with the requirement that  $\hat{x}$  is a positive vector:  $\hat{x} > [0]$ . Therefore we assume the following set of conditions:

**Assumptions (I)'**. There exist a vector  $\hat{x} > [0]$ , a price vector  $\hat{p} \geq [0]$  and a number  $\alpha > 0$  such that  $(\hat{x}, \alpha\hat{x}) \in T$ ,

- a)  $\hat{p}y - \alpha\hat{p}x \leq 0$ , for every  $(x, y) \in T$ ,
- b)  $\hat{p}y - \alpha\hat{p}x < 0$ , for every  $(x, y) \in T$ , such that  $x$  is not proporzional to  $\hat{x}$ .

Assumptions (I)', together with  $(T_5)$ , imply  $\hat{p} \geq [0]$ . Indeed, if a given component of  $\hat{p}$  would be zero, then there exists a vector  $x \geq [0]$  such that  $\hat{p}x = 0$ . From  $(T_5)$  it results the existence of a vector  $y \geq [0]$  such that  $(x, y) \in T$ . We have  $0 \leq \hat{p}y = \hat{p}(y - \alpha x) \leq 0$ . Therefore  $\hat{p}(y - \alpha x) = 0$ . This equality implies  $x$  proportional to  $\hat{x}$  and we obtain the contradiction  $\hat{p}\hat{x} = 0$ .

We now prove the strong Turnpike Theorem of Nikaido.

**Theorem 15.** Consider a technology  $T$  with assumptions  $(T_1)$ ,  $(T_2)$  and  $(T_5)$  satisfied. Let  $x^0 \in \mathbb{R}_+^n$ ,  $p \in \mathbb{R}_+^n$ . Under Assumptions (I)', if:

- a) there exists a trajectory  $\{\bar{x}(t)\}_{t=0}^\infty \in X_T(x^0)$ , with average growth rate  $\alpha - 1$ ;
- b) there exist numbers  $k' \geq k'' > 0$  such that  $k''\hat{p} \leq p \leq k'\hat{p}$ ,

then for any number  $\varepsilon > 0$  there exists a positive integer  $k_\varepsilon$  such that, for any  $N > 2k_\varepsilon$  and any trajectory  $\{x(t)\}_{t=0}^N$ ,  $p$ -optimal in  $X_T(x^0, N)$ , the inequality

$$\left\| \frac{x(t)}{\|x(t)\|} - \hat{x} \right\| < \varepsilon$$

is satisfied for all periods  $t$ , with  $k_\varepsilon \leq t \leq N - k_\varepsilon$ .

This theorem means that any optimal trajectory lies entirely in the neighbouring cone of the balanced growth trajectory in all periods extending from  $k_\varepsilon$  to  $N - k_\varepsilon$ . Without loss of generality we assume that

$$\hat{p}\hat{x} = 1.$$

It is known that any vector  $x \in \mathbb{R}^n$  can be uniquely decomposed to a sum of its orthogonal projection  $\theta(x)\hat{x}$  on the von Neumann ray  $\{\lambda\hat{x}\}_{\lambda \geq 0}$  and its orthogonal complement  $e(x)$ . Explicitly, we have

$$x = \theta(x)\hat{x} + e(x).$$

Therefore  $\hat{x}e(x) = 0$ ,  $\theta(x) = \hat{x}x$ . We shall use the notation

$$c = \min_i \hat{x}_i > 0, \tag{102}$$

$$d = \min_i \widehat{p}_i > 0. \quad (103)$$

In order to prove Theorem 15 we need the following lemmas.

**Lemma 15.** Let  $x \neq [0]$ . If

$$\left\| \frac{x}{\|x\|} - \widehat{x} \right\| < \beta$$

for a given  $\beta > 0$ , then  $\|e(x)\| < \beta \|x\|$ .

**Proof.** We have

$$\frac{x}{\|x\|} = \theta \left( \frac{x}{\|x\|} \right) \widehat{x} + e \left( \frac{x}{\|x\|} \right), \quad \widehat{x} e \left( \frac{x}{\|x\|} \right) = 0.$$

So, we get

$$\beta^2 > \left\| \frac{x}{\|x\|} - \widehat{x} \right\|^2 = \left\| \left( \theta \left( \frac{x}{\|x\|} \right) - 1 \right) \widehat{x} \right\|^2 + \left\| e \left( \frac{x}{\|x\|} \right) \right\|^2 \geq \left\| e \left( \frac{x}{\|x\|} \right) \right\|^2 = \frac{\|e(x)\|^2}{\|x\|^2}.$$

□

**Lemma 16.**

1) There is a number  $\Delta > 0$  such that the inequality

$$\left\| \frac{x(t)}{\alpha^t} \right\| \leq \Delta, \quad t = 0, 1, \dots, N$$

holds uniformly for all trajectories  $\{x(t)\}_{t=0}^N \in X_T(x^0, N)$ , for every  $N$ .

2) There is a number  $\Gamma > 0$  such that the inequality

$$\frac{\widehat{p}x(N)}{\alpha^N} \geq \Gamma$$

holds for all trajectories  $p$ -optimal in  $X_T(x^0, N)$ .

**Proof.**

1) We have

$$\frac{\widehat{p}x(t)}{\alpha^t} \leq \widehat{p}x^0.$$

Form this we obtain

$$d \left\| \frac{x(t)}{\alpha^t} \right\| \leq \frac{d \sum_{i=1}^n x_i(t)}{\alpha^t} \leq \frac{\sum_{i=1}^n \widehat{p}_i x_i(t)}{\alpha^t} = \frac{\widehat{p}x(t)}{\alpha^t} \leq \widehat{p}x^0.$$

By choosing the notation

$$\Delta = \frac{\widehat{p}x^0}{d}$$

we obtain the desired inequality.

2) Let  $\{x(t)\}_{t=0}^N \in X_T(x^0, N)$  be a  $p$ -optimal trajectory. We compare this trajectory with the trajectory  $\{\bar{x}(t)\}_{t=0}^\infty \in X_T(x^0)$ , whose existence is assured by condition a) of Theorem 15. The optimality of trajectory  $\{x(t)\}_{t=0}^N$  implies  $px(N) \geq p\bar{x}(N)$ . By condition b) of Theorem 15 we get

$$\widehat{p}x(N) \geq \frac{px(N)}{k'} \geq \frac{p\bar{x}(N)}{k'} \geq \frac{k''}{k'}\widehat{p}\bar{x}(N). \quad (104)$$

As the trajectory  $\{\bar{x}(t)\}_{t=0}^\infty$  grows at an average rate  $\alpha - 1$ , we have

$$\frac{\widehat{p}\bar{x}(N)}{\alpha^N} \geq \zeta > 0. \quad (105)$$

From (104) and (105), by denoting

$$\Gamma = \frac{k''\zeta}{k'},$$

we obtain

$$\frac{\widehat{p}x(N)}{\alpha^N} \geq \Gamma.$$

□

**Lemma 17.** Let be  $\{x(t)\}_{t=0}^N \in X_T(x^0, N)$ . If  $x(t) \neq [0]$  and

$$\left\| \frac{x(t)}{\|x(t)\|} - \widehat{x} \right\| < \beta,$$

then

$$\left\| e\left(\frac{x(t)}{\alpha^t}\right) \right\| \leq \beta\Delta.$$

**Proof.** From Lemma 15 we obtain  $\|e(x(t))\| < \beta\|x(t)\|$ . By dividing both members of this inequality by  $\alpha^t$  and taking inequality 1) of lemma 16 into account, we get the thesis.

□

**Lemma 18** Let  $\{x(t)\}_{t=0}^N$  be a  $p$ -optimal trajectory of  $X_T(x^0, N)$ . If, for some  $\eta > 0$  we have

$$\left\| e\left(\frac{x(r)}{\alpha^r}\right) \right\| < \eta \text{ and } \left\| e\left(\frac{x(s)}{\alpha^s}\right) \right\| < \eta, \quad 0 \leq r < s,$$

then

$$\theta\left(\frac{x(r)}{\alpha^r}\right) - \theta\left(\frac{x(s)}{\alpha^s}\right) < \frac{2\eta}{c}.$$

**Proof.** We put

$$\omega = \theta \left( \frac{x(r)}{\alpha^r} \right) - \theta \left( \frac{x(s)}{\alpha^s} \right).$$

Let us absurdly suppose that  $\omega \geq \frac{2\eta}{c}$ . Let

$$\widehat{\omega} = \frac{1}{2} \left[ \theta \left( \frac{x(r)}{\alpha^r} \right) + \theta \left( \frac{x(s)}{\alpha^s} \right) \right].$$

We obtain

$$\begin{aligned} \frac{\omega}{2} \widehat{x}_i &\geq \eta > \left\| e \left( \frac{x(r)}{\alpha^r} \right) \right\| \geq -e_i \left( \frac{x(r)}{\alpha^r} \right), \\ \frac{\omega}{2} \widehat{x}_i &\geq \eta > \left\| e \left( \frac{x(s)}{\alpha^s} \right) \right\| \geq -e_i \left( \frac{x(s)}{\alpha^s} \right). \end{aligned}$$

From this we have

$$\begin{aligned} \frac{x(r)}{\alpha^r} &= \theta \left( \frac{x(r)}{\alpha^r} \right) \widehat{x} + e \left( \frac{x(r)}{\alpha^r} \right) > \left( \theta \left( \frac{x(r)}{\alpha^r} \right) - \frac{\omega}{2} \right) \widehat{x} = \widehat{\omega}, \\ \frac{x(s)}{\alpha^s} &= \theta \left( \frac{x(s)}{\alpha^s} \right) \widehat{x} + e \left( \frac{x(s)}{\alpha^s} \right) < \left( \theta \left( \frac{x(s)}{\alpha^s} \right) + \frac{\omega}{2} \right) \widehat{x} = \widehat{\omega}. \end{aligned}$$

Clearly  $(\widehat{\omega}, \alpha\widehat{\omega}) \in T$ . Therefore,  $(x(r)/\alpha^r) > \widehat{\omega}$  implies, by assumption  $(T_5)$ , that  $\left( \frac{x(r)}{\alpha^r}, \alpha\widehat{\omega} \right) \in T$ . Hence,  $(x(r), \alpha^{r+1}\widehat{\omega}) \in T$ . On the other hand, from the inequality  $\frac{x(s)}{\alpha^s} < \widehat{\omega}$ , it results that we can choose a positive vector  $\widehat{v}$ , proportional to  $\widehat{x}$ , such that

$$\alpha^s \widehat{\omega} > x(s) + \alpha^s \widehat{v}.$$

But, as  $(\alpha^{s-1}\widehat{\omega}, \alpha^s\widehat{\omega}) \in T$ , assumption  $(T_5)$  implies that  $(\alpha^{s-1}\widehat{\omega}, x(s) + \alpha^s\widehat{v}) \in T$ . These results mean that the sequence  $x^0, \dots, x(r), \alpha^{r+1}\widehat{\omega}, \alpha^{r+2}\widehat{\omega}, \dots, \alpha^{s-1}\widehat{\omega}, x(s) + \alpha^s\widehat{v}, \dots, x(N) + \alpha^N\widehat{v}$  is a trajectory of  $X_T(x^0, N)$ . The inequality  $x(N) + \alpha^N\widehat{v} > x(N)$  implies  $p(x(N) + \alpha^N\widehat{v}) > px(N)$ , which contradicts the  $p$ -optimality of the trajectory  $\{x(t)\}_{t=0}^N$ . □

**Lemma 19.** Under the same assumption, of Lemma 18, we have

$$0 \leq \widehat{p} \frac{x(t)}{\alpha^t} - \widehat{p} \frac{x(t+1)}{\alpha^{t+1}} < 4\eta \max \left( \|p\|, \frac{1}{c} \right)$$

for any  $t$  such that  $r \leq t \leq s-1$ .

**Proof.** For any  $r < s$ , as  $\widehat{p}\widehat{x} = 1$ , on the ground of Lemma 17, we have

$$\begin{aligned}\widehat{p}\frac{x(r)}{\alpha^r} - \widehat{p}\frac{x(s)}{\alpha^s} &= \theta\left(\frac{x(r)}{\alpha^r}\right) - \theta\left(\frac{x(s)}{\alpha^s}\right) + \widehat{p}\left[e\left(\frac{x(r)}{\alpha^r}\right) - e\left(\frac{x(s)}{\alpha^s}\right)\right] < \\ &< 2\frac{\eta}{c} + 2\eta\|\widehat{p}\| \leq 4\eta\max\left(\|p\|, \frac{1}{c}\right).\end{aligned}$$

From this relation, for  $r \leq t \leq s - 1$ , we obtain

$$0 \leq \widehat{p}\frac{x(t)}{\alpha^t} - \widehat{p}\frac{x(t+1)}{\alpha^{t+1}} \leq \widehat{p}\frac{x(r)}{\alpha^r} - \widehat{p}\frac{x(s)}{\alpha^s} < 4\eta\max\left(\|p\|, \frac{1}{c}\right).$$

□

**Proof of Theorem 15.** On the ground of lemma 14, for any  $\varepsilon > 0$  there exists a number  $\delta_\varepsilon \in (0, 1)$  such that  $\widehat{p}y \leq (1 - \delta_\varepsilon)\alpha\widehat{p}x$  for every  $(x, y) \in T$  with

$$\left\|\frac{x}{\|x\|} - \widehat{x}\right\| \geq \varepsilon.$$

Now choose  $\eta_\varepsilon > 0$  and  $\beta_\varepsilon > 0$  such that

$$4\eta_\varepsilon\max\left(\|p\|, \frac{1}{c}\right) < \delta_\varepsilon\Gamma \quad (106)$$

$$\beta_\varepsilon\Delta < \eta_\varepsilon, \quad (107)$$

$$\beta_\varepsilon \leq \varepsilon, \quad (108)$$

where  $\Gamma$  and  $\Delta$  are the constants considered in Lemma 16.

As the assumptions of Theorem 14 are verified, there exists a positive number  $k_\varepsilon$  such that the inequality

$$\left\|\frac{x(t)}{\|x(t)\|} - \widehat{x}\right\| < \beta_\varepsilon \quad (109)$$

is satisfied, except possibly for at most  $k_\varepsilon$  periods. From the inequality 2) of Lemma 16, it results  $x(t) \neq [0]$  for all indices  $t$ . Let us suppose  $N > k_\varepsilon$  and let  $r_\varepsilon$  be, respectively, the first and the last period in which inequality (109) is satisfied. Then,

$$\left\|\frac{x(r_\varepsilon)}{\|x(r_\varepsilon)\|} - \widehat{x}\right\| < \beta_\varepsilon \quad \text{and} \quad \left\|\frac{x(s_\varepsilon)}{\|x(s_\varepsilon)\|} - \widehat{x}\right\| < \beta_\varepsilon.$$

Accordingly, in view of (106), (107) and Lemmas 17 and 19, we have, for  $r_\varepsilon \leq t \leq s_\varepsilon$

$$0 \leq \widehat{p}\frac{x(t)}{\alpha^t} - \widehat{p}\frac{x(t+1)}{\alpha^{t+1}} \leq \delta_\varepsilon\Gamma. \quad (110)$$

Suppose that for some index  $t$  between  $r_\varepsilon$  and  $s_\varepsilon - 1$  we would have

$$\left\| \frac{x(t)}{\|x(t)\|} - \hat{x} \right\| \geq \varepsilon.$$

Then,  $\hat{p}x(t+1) \leq (1 - \delta_\varepsilon)\alpha\hat{p}x(t)$ , which, together with inequality 2) of lemma 16, gives

$$\hat{p} \frac{x(t)}{\alpha^t} - \hat{p} \frac{x(t+1)}{\alpha^{t+1}} \geq \delta_\varepsilon \frac{\hat{p}x(t)}{\alpha^t} \geq \delta_\varepsilon \frac{\hat{p}x(N)}{\alpha^N} \geq \delta_\varepsilon \Gamma.$$

But this contradicts (110). Taking (108) into account, we have thereby shown that

$$\left\| \frac{x(t)}{\|x(t)\|} - \hat{x} \right\| < \varepsilon \quad (111)$$

for  $r_\varepsilon \leq t \leq s_\varepsilon$ .

From Theorem 14, we remark that surely we have  $r_\varepsilon + (N - s_\varepsilon) \leq k_\varepsilon$ . Therefore, if  $N \geq 2k_\varepsilon$ , a fortiori from (111) it results

$$\left\| \frac{x(t)}{\|x(t)\|} - \hat{x} \right\| < \varepsilon, \text{ for } k_\varepsilon \leq t \leq N - k_\varepsilon.$$

Moreover the numbers  $r_\varepsilon$ ,  $s_\varepsilon$  and  $k_\varepsilon$  are independent of the length  $N$  of the planning period.  $\square$

We now present some results concerning the asymptotic behavior of infinite trajectories.

**Theorem 16.** If the technology  $T$  satisfies, besides assumptions  $(T_1)$  and  $(T_2)$ , also assumptions  $(I)$ , then for every trajectory  $\{x(t)\}_{t=0}^\infty \in X_T(x^0)$  which grows at an average rate  $\alpha - 1$ , it holds

$$\lim_{t \rightarrow \infty} \left\| \frac{x(t)}{\|x(t)\|} - \hat{x} \right\| = 0.$$

**Proof.** We begin to evaluate in  $\mathbb{R}_+^n \times \mathbb{R}_+^n$  the distance between  $\frac{(x(t), x(t+1))}{\|x(t)\|}$  and the ray  $\{\lambda(\hat{x}, \alpha\hat{x})\}_{\lambda \geq 0}$ :

$$d\left(\frac{(x(t), x(t+1))}{\|x(t)\|}, \{\lambda(\hat{x}, \alpha\hat{x})\}_{\lambda \geq 0}\right) = \inf_{\lambda \geq 0} \left\| \frac{(x(t), x(t+1))}{\|x(t)\|} - \lambda(\hat{x}, \alpha\hat{x}) \right\|. \quad (112)$$

We note that vector  $(\alpha\hat{p}, -\hat{p})$  is orthogonal to the ray  $\{\lambda(\hat{x}, \alpha\hat{x})\}_{\lambda \geq 0}$ . Therefore the distance which does not interest will be precisely the norm of the projection of  $\frac{(x(t), x(t+1))}{\|x(t)\|}$  on  $(\alpha\hat{p}, -\hat{p})$ .

We obtain

$$\inf_{\lambda \geq 0} \left\| \frac{(x(t), x(t+1))}{\|x(t)\|} - \lambda(\hat{x}, \alpha\hat{x}) \right\| = \frac{(x(t), x(t+1))(\alpha\hat{p}, -\hat{p})}{\|x(t)\| \|(\alpha\hat{p}, -\hat{p})\|} = \frac{\alpha\hat{p}x(t) - \hat{p}x(t+1)}{\|x(t)\| \cdot \|(\alpha\hat{p}, -\hat{p})\|}. \quad (113)$$

Let  $F$  be the cone generated by the unitary vectors of the coordinates corresponding to the positive components of vector  $\hat{p}$ .

Define on  $\mathbb{R}^n$  the norm  $\|\cdot\|_1$  by the relation

$$\|x\|_1 = \|\hat{p}x\| + d(x, F). \quad (114)$$

As  $F \subset \mathbb{R}_+^n$ , each vector  $x \in \mathbb{R}_+^n$  can be uniquely written as

$$x = \tilde{x} + \tilde{\tilde{x}},$$

where  $\tilde{x} \in F$  is obtained from  $x$ , by putting equal to zero those components which correspond to a zero component of  $\hat{p}$ , and  $\tilde{\tilde{x}}$  is obtained from  $x$ , by putting equal to zero the remaining components. Obviously  $\tilde{x}\tilde{\tilde{x}} = 0$ . From this, for  $x \in \mathbb{R}_+^n$  we obtain:

$$\|x_1\| = \hat{p}(\tilde{x} + \tilde{\tilde{x}}) + d(\tilde{x} + \tilde{\tilde{x}}, F) = \hat{p}\tilde{x} + d(\tilde{\tilde{x}}, F) = \hat{p}\tilde{x} + \|\tilde{\tilde{x}}\| = \|\tilde{x}\|_1 + \|\tilde{\tilde{x}}\|_1. \quad (115)$$

Taking (115) into account, we have

$$0 \leq \frac{\alpha\hat{p}x(t) - \hat{p}x(t+1)}{\|x(t)\|_1} = \left( \alpha - \frac{\hat{p}x(t+1)}{\hat{p}x(t)} \right) \left( \frac{\|\tilde{x}(t)\|_1}{\|\tilde{x}(t)\|_1 + \|\tilde{\tilde{x}}(t)\|_1} \right) \leq \alpha - \frac{\hat{p}x(t+1)}{\hat{p}x(t)}. \quad (116)$$

As the trajectory  $\{x(t)\}_{t=0}^\infty$  grows at the average rate  $\alpha - 1$  we have

$$\lim_{t \rightarrow \infty} \left( \alpha - \frac{\hat{p}x(t+1)}{\hat{p}x(t)} \right) = 0,$$

so that, from (113) and (116) we deduce

$$\lim_{t \rightarrow \infty} \inf_{\lambda \geq 0} \left\| \frac{(x(t), x(t+1))}{\|x(t)\|} - \lambda(\hat{x}, \alpha\hat{x}) \right\| = 0. \quad (117)$$

For all indices  $t$  the following relation is verified:

$$\inf_{\lambda \geq 0} \left\| \frac{(x(t), x(t+1))}{\|x(t)\|} - \lambda(\hat{x}, \alpha\hat{x}) \right\| = \inf_{\lambda \geq 0} \left\| \left( \frac{x(t)}{\|x(t)\|} - \lambda\hat{x}, \frac{x(t+1)}{\|x(t)\|} - \lambda\alpha\hat{x} \right) \right\| \geq \inf_{\lambda \geq 0} \left\| \frac{x(t)}{\|x(t)\|} - \lambda\hat{x} \right\|,$$

and there exists a number  $\lambda(t) \geq 0$  such that

$$\inf_{\lambda \geq 0} \left\| \frac{x(t)}{\|x(t)\|} - \lambda \hat{x} \right\| = \left\| \frac{x(t)}{\|x(t)\|} - \lambda(t) \hat{x} \right\|.$$

It results

$$\lim_{t \rightarrow \infty} \left\| \frac{x(t)}{\|x(t)\|} - \lambda(t) \hat{x} \right\| = 0. \quad (118)$$

In conclusion, the following inequalities

$$\begin{aligned} \left\| \frac{x(t)}{\|x(t)\|} - \hat{x} \right\| &= \left\| \frac{x(t)}{\|x(t)\|} - \lambda(t) \hat{x} - (1 - \lambda(t)) \hat{x} \right\| \leq \left\| \frac{x(t)}{\|x(t)\|} - \lambda(t) \hat{x} \right\| + |1 - \lambda(t)| = \\ &= \left\| \frac{x(t)}{\|x(t)\|} - \lambda(t) \hat{x} \right\| + \left| \left\| \frac{x(t)}{\|x(t)\|} \right\| - \lambda(t) \right| \leq 2 \left\| \frac{x(t)}{\|x(t)\|} - \lambda(t) \hat{x} \right\|, \end{aligned}$$

together with (118), give

$$\lim_{t \rightarrow \infty} \left\| \frac{x(t)}{\|x(t)\|} - \hat{x} \right\| = 0.$$

□

**Theorem 17.** If the technology  $T$  satisfies assumption  $(T_1)$ ,  $(T_2)$  and  $(I)$  and if  $\hat{p} > [0]$ , then for every trajectory  $\{x(t)\}_{t=0}^{\infty} \in X_T(x^0)$  there exists a number  $\xi \geq 0$  such that

$$\lim_{t \rightarrow \infty} \frac{x(t)}{\alpha^t} = \xi \hat{x}. \quad (119)$$

**Proof.** As  $\hat{p} > [0]$ , we have  $F = \mathbb{R}_+^n$  and therefore  $\|x\|_1 = \hat{p}x$  for  $x \in \mathbb{R}_+^n$ . If the trajectory  $\{x(t)\}_{t=0}^{\infty}$  does not grow at an average rate  $\alpha - 1$ , then

$$\lim_{t \rightarrow \infty} \frac{\|x(t)\|_1}{\alpha^t} = \lim_{t \rightarrow \infty} \frac{\hat{p}x(t)}{\alpha^t} = 0$$

and (119) is verified with  $\xi = 0$ .

If the trajectory  $\{x(t)\}_{t=0}^{\infty}$  grows at an average rate  $\alpha - 1$ , then, by Theorem 16:

$$\lim_{t \rightarrow \infty} \frac{x(t)}{\|x(t)\|_1} = \hat{x}. \quad (120)$$

Relation (120) can be rewritten as

$$\lim_{t \rightarrow \infty} \frac{x(t)}{\hat{p}x(t)} = \hat{x}. \quad (121)$$

Denoting

$$\lim_{t \rightarrow \infty} \frac{\hat{p}x(t)}{\alpha^t} = \xi > 0,$$

from (121) we have

$$\lim_{t \rightarrow \infty} \frac{x(t)}{\alpha^t} = \xi \widehat{x}.$$

□

**Theorem 18.** Let the technology  $T$  satisfy assumptions  $(T_1)$ ,  $(T_2)$ ,  $(T_3)$ ,  $(T_4)$  and  $(I)'$ . Let  $x^0 > [0]$ . If  $\{x(t)\}_{t=0}^{\infty} \in X_T(x^0)$  is an intertemporally efficient trajectory, then

$$\lim_{t \rightarrow \infty} \left\| \frac{x(t)}{\|x(t)\|} - \widehat{x} \right\| = 0.$$

**Proof.** Under the assumptions of the theorem, from Theorem 12 it follows that the trajectory  $\{x(t)\}_{t=0}^{\infty}$  grows at the average rate  $\alpha - 1$ . Then, by Theorem 16 we obtain the thesis.

□

**Remark 12.** The Radner-Nikaido analysis on turnpike properties, as Radner (1961) explicitly recognized (due to the assumption that  $\widehat{p}(y - \alpha x) < 0$  for all  $(x, y) \in T$  such that  $x$  is not proportional to  $\widehat{x}$ ), does not apply to the production set which is a polyhedral cone, i. e. to the original von Neumann model.

Morishima (1964) presents a generalization of the Radner-Nikaido theorem that does not suffer from this limitation (obviously the assumptions of Morishima differ in part from the ones of Radner and Nikaido). Turnpikes theorems specifically built for the original (polyhedral) von Neumann model are presented, e. g., by McKenzie (1967), Morishima (1969) and Ashmanov (1983). The latter author considers a matrix pair  $(A, B)$ , with matrices  $A$  and  $B$  not necessarily nonnegative, whereas McKenzie (1967) proves his turnpike results by means of the concept of *von Neumann facet*. In general, let the technology  $T$  satisfy assumption  $(T_1)$ ,  $(T_2)$ ,  $(T_3)$  and  $(T_4)$ . Let  $P$  be the set of the von Neumann price vectors of the technology. For every  $p \in P$  we denote

$$H_p = \{(x, y) : (x, y) \in T, \alpha(T)px = py\}.$$

The set  $F^* = \bigcap_{p \in P} H_p$  is called *von Neumann facet* of the technology  $T$ . It can be shown that  $F^*$  is a closed convex cone and that for every  $p \in \text{relint}(P)$ , it holds  $F^* = H_p$ .

See also McKenzie (1963a). Under the said assumptions, and for a general Gale-von Neumann model, it is possible to prove the following result (McKenzie (1963a, 1967) uses the so-called “absolute norm” of a vector  $z \in \mathbb{R}^n : \|z\|_a = \sum |z_i|$ , the sum of the absolute values of its components, so that the angular distance between two nonzero vectors  $z$  and  $w$  becomes  $d(z, w) = \sum |z_i| / \|z\|_a - w_i / \|w\|_a$ ). The angular distance of a vector  $z$  and a set of vectors  $C$  is defined by  $d(z, C) = \inf d(z, w)$  for all  $w \in C$ ).

Let  $\widehat{p} \in \text{relint}(P)$ ,  $x^0 \in \mathbb{R}_+^n$ ,  $p \in \mathbb{R}_+^n$ . If

- a) There exists a trajectory  $\{\bar{x}(t)\}_{t=0}^{\infty} \in X_T(x^0)$  which grows at the average rate  $\alpha(T) - 1$ ;

b) There exist numbers  $K' \geq k'' > 0$  such that  $k'' \hat{p} \leq p \leq k' \hat{p}$ ;

then for every  $\varepsilon > 0$  there exists a positive integer  $k_\varepsilon$  such that for every  $N$  and for every  $p$ -optimal trajectory  $\{x(t)\}_{t=0}^N \in X_T(x^0, N)$ , the number of processes  $(x(t), x(t+1))$  such that

$$d\left(\frac{(x(t), x(t+1))}{\|x(t)\|}, F^*\right) \geq \varepsilon$$

cannot exceed  $k_\varepsilon$ .

## References

- S. A. ASHMANOV (1983), An Algebraic approach to the proof of turnpike theorems, *U.S.S.R. Comput. Maths. Math. Phys.*, 23, 18-22.
- S. CRUCEANU (1978), *Introducere in Teoria Matematica a Modelelor de Crestere de Tip von Neumann*, Editura Academiei Republicii Socialiste Romania, Bucuresti.
- G. DEBREU (1951), The coefficient of resource utilization, *Econometrica*, 19, 273-292.
- R. DORFMAN, P. A. SAMUELSON and R. M. SOLOW (1958), *Linear Programming and Economic Analysis*, McGraw-Hill, New York.
- E. M. DRANDAKIS (1966), On efficient accumulation paths in the closed production model, *Econometrica*, 34, 331-346.
- I. V. EVSTIGNEEV and K. R. SHENK-HOPPÉ (2006), The von Neumann-Gale growth model and its stochastic generalization; in R.-A. Dana, C. Le Van, T. Mitra and K. Nishimura (Eds.), *Handbook of Optimal Growth, Vol. I*, Springer, New York, 337-383.
- F. M. FISHER (1963), Balanced growth and intertemporal efficiency in capital accumulation: comment, *Int. Economic Rev.*, 4, 232-234.
- H. FURUYA and K. INADA (1962), Balanced growth and intertemporal efficiency in capital accumulation, *Int. Economic Rev.*, 3, 94-107.
- D. GALE (1956), The closed linear model of production; in H. W. Kuhn and A. W. Tucker (Eds.), *Linear Inequalities and Related Systems*, Princeton Univ. Press, Princeton, 285-303.
- D. GALE (1960), *The Theory of Linear Economic Models*, McGraw-Hill Book Co., New York.
- D. GALE (1967), A geometric duality theorem with economic applications, *Rev. Econ. Studies*, 34, 19-24.
- D. GALE (1972), Comment, *Econometrica*, 40, 391-392.
- A. M. GEOFFRION (1971), Duality in nonlinear programming: a simplified applications-oriented development, *SIAM Rev.*, 13, 1-37.

- G. GIORGI (2004), Modelli di von Neumann non poliedrici, *Rapporto N. 24 A.M.A.S.E.S., CUEN, Napoli*.
- G. GIORGI and M. R. MERIGGI (1987), Structural properties of von Neumann and related models, *Atti XI Convegno A.M.A.S.E.S., Pitagora, Bologna*, 301-349.
- G. GIORGI and M. R. MERIGGI (1988), On the hierarchy of some qualitative structures in the von Neumann and related models, *Metroeconomica*, 39, 121-140.
- D. GLYCOPANTIS (1970), The closed linear model of production: a note, *Rev. Econ. Studies*, 37, 295-297.
- C. H. HOWE (1960), An alternative proof of the existence of general equilibrium in a von Neumann model, *Econometrica*, 28, 635-639.
- J. HULSMANN and V. STEINMETZ (1972), A note on the non existence of optimal price vectors in the general balanced-growth model of Gale, *Econometrica*, 40, 387-389.
- K. INADA (1964), Some structural characteristics of turnpike theorems, *Rev. Economic Studies*, 31, 43-58.
- S. KARLIN (1959), *Mathematical Methods and Theory in Games, Programming and Economics, I, II*, Addison-Wesley, Reading, Mass.
- J. KEMENY, O. MORGENSTERN and G. THOMPSON (1956), A generalization of the von Neumann model of an expanding economy, *Econometrica*, 24, 115-135.
- J. LOS (1971), A simple proof of the existence of equilibrium in a von Neumann model and some of its consequences, *Bull. de l'Académie Polonaise des Sciences, Séries des Sciences Math., Astr. et Phys.*, 19, 971-979.
- E. MALINVAUD (1953), Efficient capital accumulation and efficient allocation of resources, *Econometrica*, 21, 233-268.
- E. MALINVAUD (1962), Efficient capital accumulation: a corrigendum, *Econometrica*, 30, 570-573.
- V. L. MAKAROV and A. M. RUBINOV (1970), Superlinear point-set maps and models of economic dynamics, *Russian Math. Surveys*, 25, 125-169.
- V. L. MAKAROV and A. M. RUBINOV (1977), *Mathematical Theory of Economic Dynamics and Equilibria*, Springer-Verlag, New York.
- L. W. McKENZIE (1963a), Turnpike theorems for a generalized Leontief model, *Econometrica*, 31, 165-180.
- L. W. McKENZIE (1963b), The Dorfman-Samuelson-Solow turnpike theorem, *Int. Economic Review*, 4, 29-43.

- L. W. McKENZIE (1967), Maximal paths in the von Neumann model; in E. Malinvaud and M. O. L. Bacharach (Eds.), *Activity Analysis in the Theory of Growth and Planning*, MacMillan, London, 43-63.
- L. W. McKENZIE (1976), Turnpike theory, *Econometrica*, 44, 841-865.
- L. W. McKENZIE (1986), Optimal economic growth, turnpike theorems and comparative dynamics: in K. J. Arrow and M. D. Intriligator (Eds.), *Handbook of Mathematical Economics*, Vol. III, Elsevier Science B. V., Amsterdam, 1281-1355.
- L. McKENZIE and M. YANO (1980), Turnpike theory: some corrections, *Econometrica*, 48, 1839-1840.
- M. MORISHIMA (1961), Proof of a turnpike theorem: the “no joint production” case, *Rev. of Economic Studies*, 28, 89-97.
- M. MORISHIMA (1964), *Equilibrium, Stability and Growth*, Clarendon Press, Oxford.
- M. MORISHIMA (1969), *Theory of Economic Growth*, Oxford Univ. Press, Oxford.
- P. C. NICOLA (1976), *Lezioni di Dinamica Economica*, Il Mulino, Bologna.
- H. NIKAIDO (1964), Persistence of continual growth near the von Neumann ray: a strong version of the Radner turnpike theorem, *Econometrica*, 32, 151-162.
- H. NIKAIDO (1968), *Convex Structures and Economic Theory*, Academic Press, New York.
- T. RADER (1975), “Turnpike” theory with (under) consumption, *Rev. of Economic Studies*, 42, 155-165.
- T. RADER (1976), “Turnpike” theory: a revised proof, *Rev. of Economic Studies*, 43, 369-370.
- R. RADNER (1961), Paths of economic growth that are optimal with regard only to final states: a turnpike theorem, *Rev. of Economic Studies*, 28, 98-104.
- R. T. ROCKAFELLAR (1970), *Convex Analysis*, Princeton Univ. Press, Princeton.
- A. L. SOYSTER (1974), The existence of optimal price vectors in the general balanced-growth model of Gale, *Econometrica*, 42, 199-205.
- A. TAKAYAMA (1985), *Mathematical Economics*, Cambridge Univ. Press, Cambridge.
- J. TSUKUI (1967), The consumption and the output turnpike theorems in a von Neumann type of model. A finite term problem, *Rev. of Economic Studies*, 34, 85-93.
- S. J. TURNOVSKY (1970), Turnpike theorems and efficient economic growth; Chapter 10 in E. Burmeister and A. R. Dobell, *Mathematical Theories of Economic Growth*, MacMillan, New York, 311-351.
- J. Von NEUMANN (1945-46), A model of general economic equilibrium, *Rev. of Economic Studies*, 13, 1-9.
- Y. WAN (1971), *Economic Growth*, Harcourt Brace Jovanovich, New York.

S. G. WINTER (1965), Some properties of the closed linear model of production, *Int. Economic Rev.*, 6, 199-210.

S. G. WINTER (1967), The norm of a closed technology and the straight-down-the-turnpike theorem, *Rev. of Economic Studies*, 34, 67-84.