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Two Proofs of the Farkas-Minkowski Theorem by Tandem Method and by Use of an Orthogonal Complement

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TWO PROOFS OF THE FARKAS-MINKOWSKI THEOREM BY A TANDEM METHOD AND BY USE OF AN ORTHOGONAL COMPLEMENT

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ABSTRACT

This note presents two proofs of the Farkas-Minkowski theorem. The first one is analytical, and this does not presuppose the closedness of a finitely generated cone. We do not employ separation theorems either. Even the concept of linear independence or invertibility of matrices is not necessary. Our new device consists in proving the Farkas-Minkowski theorem and the closedness of a finitely generated cone at the same time based upon mathematical induction. We make use of a distance minimization problem with an equality constraint. The second proof is algebraic, and a mixture of Gale's and Ben-Israel's methods. Our proof based on the orthogonal complement seems easy to understand in terms of geometrical images.

1. INTRODUCTION

There have already been many proofs of the Farkas-Minkowski theorem or Farkas's lemma concerning the existence of a nonnegative solution for a system of linear equations. The reader is referred to Giorgi (2007). The proofs can be classified into two groups, topological and algebraic. Many of the former group use one of separation theorems, and the theorem has been extended from finite dimensional Euclidean spaces to infinite dimensional locally convex topological spaces. See, for example, Hurwicz (1958), Braunschweiger and Clark (1962), and Ben-Israel and Charnes (1968). More than five decades ago, an interesting proof was provided by Dorfman, Samuelson and Solow (1958), which used a simple minimization problem on the domain of the nonnegative orthant of the Euclidean space. Morishima (1969) adopted the same method more explicitly by using differential calculus.

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The same norm minimization method was employed in Dax (1997) without assuming, however, the closedness of a finitely generated cone. A rigorous proof based on this minimization method is explained in Matoušek and Gärtner (2007) together with a proof that a finitely generated cone is closed. This proof needs the concepts of linear independence and the invertibility of a matrix.

The second group of algebraic proofs can allow for fields more general than reals over which a given vector space is defined. Gordan (1873) discussed about the solvability in integers when he published an algebraic proof of Gordan's theorem on nonnegative real solutions. Charnes and Cooper (1958) provided an elementary half-algebraic and half-geometric proof based upon such basic concepts as linear independence and Cramer's rule. In Ben-Israel (1964), he obtained Tucker's key theorem by use of orthogonal complements. Within this group, are the proofs based on the Fourier-Motzkin elimination, which might be made an independent third group of purely algorithmic proofs. The proofs are simple as well as elementary. The reader is referred to Fourier (1890), Motzkin (1951) and Matoušek and Gärtner (2007, pp. 100-104).

It seems, however, that many authors as well as teachers concerned with optimization theory are not fully satisfied with their presentation of the Farkas-Minkowski theorem, either by their exclusion of a proof concerning the closedness of a finitely generated cone, or their inclusion of one of its lengthy proofs, or, in the case of algebraic proofs, the lacking in 'leading lights' through their proofs. In a footnote, Dorfman, Samuelson and Solow (1958) wrote on their omitted proof: "At this point we sweep a technical difficulty under the rug. How do we know that there is a vector in the cone at minimum distance from c? The assertion sounds plausible and is also true; but, since the proof is rather detailed, we leave the statement with an appeal as to its plausibility." Gill, Murray and Wright (1991) added about their detailed proof: "The reader who is willing to accept the entire lemma on faith can simply skip to ..." On the other hand, about his algebraic proof, Gale (1960) noted: "unfortunately, the proof is rather formal and does not make clear why the theorem 'works'." (It is remembered, however, that not all the mathematicians agree with this view of Gale, as Bartl (2012a) notes Bland's opinion. Certainly, who gets what sort of geometrical images from a given proof can vary considerably.) Though recent algebraic proofs are short enough, they are, however, not so easy to grasp geometrically, and when teaching in a lecture room without a projector, a memo is indispensable. And yet most math teachers are too busy to work out their own satisfactory proof.

Borwein (1983) observed the equivalence of the Farkas-Minkowski theorem and the closedness of a finitely generated cone, implying both can be a pedagogical annoyance as was remarked in Gill, Murray and Wright (1991). On the other hand, Tucker's theorem on linear systems in Tucker (1956), which Good (1959) calls the key theorem in the treatment of linear inequalities, is also known to be equivalent to the Farkas-Minkowski theorem. One of the authors, Fujimoto (1976) published a simple proof of Tucker's theorem by use of a constrained minimization problem. A question is whether it is possible to adapt Fujimoto's proof to get the Farkas-Minkowski theorem directly without requiring the closedness of a finitely generated cone. The answer seems to be in the affirmative, and this note is to report this as our first new proof, which will show the Farkas-Minkowski theorem itself together with the closedness of a finitely generated cone at a stroke. After all, it is desirable to show, at some time during the course to the students, the closedness of a finitely generated cone. We depend less on linear algebra dispensing with the use of linear independence nor matrix invertibility, and more on elementary calculus, i.e., Lagrange's multiplier method with nonnegative constraints, which is a technique familiar to students of economics, not to mention those of mathematics.

One of the recent short algebraic proofs is in Bartl (2012b), which can deal also with the vector spaces over the field of rationals, or more general fields as was suggested in a classical paper by Gordan (1873). (See also Charnes and Cooper (1958), Good (1959), Conforti, Di Summa and Zambelli (2007), and Bartl (2008, 2012a). Bartl's proof in Bartl (2008) was explained in detail by Jaćimović (2011) for the finite dimensional case, and by Perng (2012).) Bartl's method of proof is a restatement of the proof of Tucker's lemma in Tucker (1956), and he considered a vector space of an arbitrary dimension over a totally ordered field with a finite number of linear functions on the vector space. His proofs are so short that they are almost satisfactory for classroom presentation. Gale's sigh might, however, be heard again because Bartl's feat looks like a remarkable tightrope act as Gale's does. Thus, our second proof, which is algebraic, is an adaptation of Gale's as in Tucker (1956), and proves Tucker's key theorem first, using an orthogonal complement in an explicit way as in Ben-Israel (1964). One important difference from the existing algebraic proofs such as Gale (1960) or Tucker (1956) is that we consider the orthogonal complement to a subspace spanned by a set of columns corresponding to positive entries in a nonnegative nonzero solution. So, our proof might allow for somewhat intuitive geometric interpretations of what we are doing within.

In the next section we explain our notation, state the Farkas-Minkowski theorem and two preliminary propositions, then we present an analytic proof in section 3. Section 4 contains an algebraic proof of Tucker's key theorem, with final section 5 giving some remarks. In the appendix, we prove in a 'naive' way one of the preliminary propositions which is concerned with the Lagrange's multiplier method with the nonnegativity constraints on variables.

2. NOTATION AND PRELIMINARIES

Let m and n be natural numbers, and $\mathbb{R}^n$ be the n-dimensional real Euclidean space, and $\mathbb{R}^n_+$ be the nonnegative orthant of $\mathbb{R}^n$. The inner-product of two vectors, v and w in $\mathbb{R}^m$, is denoted by $v' \cdot w$, and the premultiplication of a vector $y \in R^m$ with an $m \times n$ matrix M is written as $y' \cdot M$ or simply $y'M$ when avoiding too many dots. The symbol S^{n-1} stands for the $(n-1)$-simplex, i.e., $S^{n-1} \equiv \{x \mid x \in \mathbb{R}^n_+, \sum_{j=1}^n x_j = 1\}$. In vector comparison, the inequality symbol in $x > y$, signifies that a strict inequality holds in each elementwise comparison. In this note, we define the *finitely generated cone* by M as the set $\{Mx \mid x \in \mathbb{R}^n_+\}$.

Let us first write down

Farkas-Minkowski theorem. Let A be a given $m \times n$ real matrix, and $b \in \mathbb{R}^m$. Then, the system of linear equations $Ax = b$ has a solution $x^* \in \mathbb{R}^n_+$ if and only if $y' \cdot A \leq 0$ for $y \in \mathbb{R}^m$ implies $y' \cdot b \leq 0$.

About an economic interpretation of the above theorem, the reader is referred to Filippini and Filippini (1982) or Fujimoto and Krause (1988).

We make use of the following two propositions. The proof of the first proposition is given in the text, while that for the second is presented in the appendix, and shows we do not need the Farkas-Minkowski theorem to get Proposition 2. Proposition 2 is important in economics, but has seldom been proved in textbooks, although it is stated and explained. See, e.g., Dixit (1990, pp.24-29). For the mathematically oriented, Proposition 2 is a corollary of the Karush-Kuhn-Tucker theorem, thus comes out logically after the Farkas-Minkowski theorem, with some struggle through the constraint qualifications. See Mas-Colell, Whinston and Green (1995, p.959-961)

Proposition 1. The Farkas-Minkowski theorem implies the closedness of a finitely generated cone.

Proof. Let the finitely generated cone by A be denoted by C, and choose a $b \in \mathbb{R}^m \setminus C$. Then, by the Farkas-Minkowski Theorem, there exists a $y \in \mathbb{R}^m$ such that $y'A \leq 0$ and $y'b > 0$. Hence the open halfspace $G \equiv \{w \in \mathbb{R}^m \,|\, y'w > \frac{1}{2}y'b\}$ does not intersect the cone C. The set G is open and $b \in G$. This proves that the complement $\mathbb{R}^m \setminus C$ of the cone C is an open set, so that the cone C is closed. □

Proposition 1 is the easier half of the equivalence result in Borwein (1983).

Proposition 2. (Lagrange's multiplier method with nonnegative constraints) Let $f(x)$ be a quadratic function with $x \in \mathbb{R}^n$. Consider the minimization problem:

$$\min\ f(x) \text{ subject to } x \in \mathbb{R}^n_+, \text{ and}$$
$$\sum_{j=1}^{r} x_j = 1, \text{ with } 1 \leq r \leq n.$$

If the minimum is attained at x^*, then there exists a real number λ such that

$$\left.\frac{\partial f(x)}{\partial x_j}\right|_{x=x^*} \geq \lambda, \text{ for } j = 1, \dots, r,$$
$$\left.\frac{\partial f(x)}{\partial x_j}\right|_{x=x^*} \cdot x_j^* = \lambda \cdot x_j^*, \text{ for } j = 1, \dots, r,$$
$$\left.\frac{\partial f(x)}{\partial x_j}\right|_{x=x^*} \geq 0, \text{ for } j = r+1, \dots, n,$$
$$\left.\frac{\partial f(x)}{\partial x_j}\right|_{x=x^*} \cdot x_j^* = 0, \text{ for } j = r+1, \dots, n.$$

3. A NEW ANALYTIC PROOF

Our method of proof is a sort of 'tandem riding' with the Farkas-Minkowski theorem and the closedness of a finitely generated cone reached at the same time, and is constructed using mathematical induction.

Theorem. The Farkas-Minkowski theorem and the closedness of a finitely generated cone both hold good.

Proof. We employ mathematical induction on n, the number of columns of a given matrix A. When $n = 1$, the Farkas-Minkowski theorem hence the closedness (by Proposition 1) is obvious.

Now we suppose these two for the case of the number of columns less than or equal to $n-1$. We prove the Farkas-Minkowski theorem for the case n. Since the 'only if' part is evident, we prove the 'if' part. Define an $m \times (n+1)$ matrix B and a vector u as

$$\begin{aligned} B &\equiv (A \quad -b), \\ u &\equiv \begin{pmatrix} x \\ z \end{pmatrix}, \; x \in \mathbb{R}^n_+, \; z \in \mathbb{R}_+, \end{aligned}$$

and let us consider the equation

$$Bu = (A \quad -b)\begin{pmatrix} x \\ z \end{pmatrix} = Ax - bz = 0.$$

If this equation has a solution $u \geq 0$ with $z \neq 0$, then we have $A(x/z) = b$. Next, suppose that this equation has a solution $u \geq 0$ with $z = 0$, but $x > 0$. In this case the set $\{Aw \mid w \in \mathbb{R}^n_+\}$ is equal to the subspace $\{Aw \mid w \in \mathbb{R}^n\}$, thus closed. If this subspace includes the vector b, again we have $Aw = b$. When the subspace does not contain b, we obtain a contradiction to the hypothesis, which is clear by the existence of a normal vector from the vector b to the subspace. That is, from the given condition, the inner-product between this normal vector and b should be zero, while it cannot be zero unless b is in the subspace. So, the remaining case is where we cannot find any solution vector $u \geq 0$ such that $x > 0$ with $z = 0$. In this case, there exists a nonnegative solution vector $\overline{x}$ to $Ax = 0$ with the minimum number of zeros among possible solutions. Let this minimum number be denoted as $r \geq 1$. It may happen $r = n$, implying the zero is the only nonnegative solution to $Ax = 0$. Without loss of generality, we assume the first r elements of $\overline{x}$ are zero. We now consider the following minimization problem:

$$\min \frac{1}{2} u' B' B u \; \text{ subject to } \; \sum_{j=1}^{r} x_j + z = 1, \; u \geq 0.$$

The set $D \equiv \{Bu \mid \sum_{j=1}^{r} x_j + z = 1, \; u \geq 0\}$ is closed, because this is actually the Minkowski sum of the compact set $(A_r, \; -b)u_r, \; u_r \in S^r$ (because the continuous image of the compact set S^r is compact), and the closed set $A_{(r)}x_{(r)}, \; x_{(r)} \in \mathbb{R}^{n-r}_+$, where A_r is the $m \times r$ matrix consisting of the first r columns of A, and $A_{(r)}$ the $m \times (n-r)$ matrix made from A by deleting the first r columns, with x_r and $x_{(r)}$ defined in a similar way. The closedness of the latter set $A_{(r)}x_{(r)}, \; x_{(r)} \in \mathbb{R}^{n-r}_+$, follows from the inductive hypothesis that a finitely generated cone is closed when the number of columns of A is less than or equal to $(n-1)$ because $r \geq 1$. Therefore the above minimization problem has the minimum at a vector $u^* \equiv (x^*,\, z^*)'$, because we can consider only those vectors in the compact set $D \cap \{y \mid \|y\| \le \|\overline{y}\|\,,\, y \in \mathbb{R}^m\}$ for an arbitrarily chosen $\overline{y} \in D$. Applying Proposition 2 we have

$$u^{*\prime} B' A \ge \lambda e_r, \qquad (1)$$
$$u^{*\prime} B' A x^* = \lambda e_r x^*, \qquad (2)$$
$$u^{*\prime} B'(-b) \ge \lambda, \qquad (3)$$
$$u^{*\prime} B'(-b) z^* = \lambda z^*, \qquad (4)$$

where $e_r \equiv (1, \dots, 1, 0, \dots, 0) \in \mathbb{R}^n$, i.e., the first r entries are unity with the remaining elements being zero. It follows from eqs. (2) and (4) above that the minimum $u^{*\prime} B' B u^* = \lambda \ge 0$.

First suppose $\lambda > 0$. Then from (1) we get $y' \cdot A \ge 0$, while we have $y' \cdot b < 0$ from (3), by putting $y \equiv Bu^*$, i.e., $y' = u^{*\prime} B'$. A contradiction to the 'if' part supposition. Hence $\lambda = 0$. When $z^* \neq 0$, the system reduces again to the very first case because $\lambda = 0$ means $Bu^* = 0$. On the other hand, when $z^* = 0$, we have $\sum_{j=1}^{r} x_j^* = 1$. The sum $x^\circ \equiv \overline{x} + x^*$ has a less number of zeros than $\overline{x}$ while satisfying $Ax^\circ = 0$, a contradiction. This means that the case with $\lambda = 0$ and $z^* = 0$ cannot take place, which ends the proof of the Farkas-Minkowski theorem for the case n.

Once we get the Farkas-Minkowski theorem for the case n, the closedness of a finitely generated cone for this case follows from Proposition 1. Hence we have shown what to be proved. □

4. AN ALGEBRAIC PROOF

Now we present an algebraic proof of Tucker's key theorem by use of mathematical induction and a method in Ben-Israel (1964), i.e., employing the orthogonal complement. Gale (1960), Tucker (1956), and Bartl (2008, 2012a) have used the orthogonal complement implicitly, and this complement is constructed against the subspace spanned by a single vector whose existence is guaranteed by the inductive hypothesis. Their way of use of orthogonal complement has made the proof a little difficult to grasp geometrically, albeit with a marvellous effect that the proof does not require the inner-product. In our proof, we consider the complement to the subspace spanned by those column vectors of A whose positive combination can form the zero vector, actually the columns which correspond the positive entries of $\overline{x}$, i.e., those columns in $A_{(r)}$, with the minimum number of zeros in the proof of the preceding section. We use the same symbols as in our analytic proof above, and continue to employ the field of real numbers.

We now prove

Tucker's key theorem. Let A be a given $m \times n$ matrix of reals. Then, the system of linear equations $Ax = 0$ for $x \in \mathbb{R}^n_+$, and the system of inequalities $y' \cdot A \ge 0$ for $y \in \mathbb{R}^m$ have a pair of solutions x^* and y^* such that $x^* + A'y^* > 0$.

Proof. We use mathematical induction on n, the number of columns of A. When $n = 1$, it is obvious. Suppose the theorem is true up to $n - 1$, and let us prove the case when the number of columns is n. First, when there exists a strictly positive solution, $x^* > 0$, to $Ax = 0$, then we can choose $y^* = 0$. Next, as in the analytic proof above, we consider a nonnegative vector $\overline{x}$ such that $A\overline{x} = 0$, and $\overline{x}$ has the minimum number of zeros among the solutions to the equation. Let us suppose the minimum number of zeros is r, $r \geq 1$, and the first r elements of $\overline{x}$ are zeros, with the remaining elements being all positive. We consider the subspace W of dimension k spanned by the last $(n-r)$ columns, i.e., the columns of $A_{(r)}$, the same symbol as in our analytic proof. It is to be noted that the subspace W can be written as $W \equiv \{A_{(r)}x_{(r)} \,|\, x_{(r)} \in \mathbb{R}^{n-r}_{+}\} = \{A_{(r)}x_{(r)} \,|\, x_{(r)} \in \mathbb{R}^{n-r}\}$. We deal with the two cases separately; the first one in which $r < n$, and the second case where $r = n$.

In the first case, we can suppose that $k \geq 1$, because $k = 0$ implies the the last $(n-r)$ columns are all zero vectors, and the case reduces to the r case. Now let us build up the orthogonal complement $W^{\perp}$ of dimension $(m-k)$ to the subspace W. We know that the space $\mathbb{R}^m$ is represented by a direct sum, that is, $\mathbb{R}^m = W + W^{\perp}$ and $W \cap W^{\perp} = \{0\}$. Each of the column vectors in A_r is then decomposed into the two parts in W and $W^{\perp}$ respectively, and we pick up the components belonging to the latter, forming a new $m \times r$ matrix $A_r^{\perp}$. Now there should be no nonzero nonnegative solution to the equation $A_r^{\perp} \cdot x_r = 0$. (Otherwise, we could get $A_r x_r^* = 0 + w$, where x_r^* is a supposed solution and w is a certain vector in W. That is, $A_r x_r^* + (-w) = 0$ for a $(-w) \in W$, which in turn implies the existence of a nonnegative vector $\widetilde{x}$ such that $A\widetilde{x} = 0$ and $\widetilde{x}$ has at least one positive element in the first r entries. We create a vector $\widetilde{x} + \overline{x}$, which would be a solution to $Ax = 0$ and have a less number of zeros than $\overline{x}$, a contradiction.) Hence, by the inductive hypothesis (actually its special case corresponding to Gordan's theorem), there exists a $y \in \mathbb{R}^m$ which satisfies $y' \cdot A_r^{\perp} > 0$. Decompose this y to the two parts in W and $W^{\perp}$ respectively, i.e., $y = y_W + y_{W^{\perp}}$. It is evident that we still have $y'_{W^{\perp}} \cdot A_r > 0$, and that $y'_{W^{\perp}} \cdot A_{(r)} = 0$, thus yielding $\overline{x} + A' y_{W^{\perp}} > 0$.

It remains for us to discuss the second case $r = n$, that is, when there is no nonzero nonnegative solution to $Ax = 0$. In such a case, there should be no zero vector among the columns of A, because $r = n$. We consider again two cases; (i) when at least one column i of A is represented as $\mathbf{a}_i = \sum_{j \neq i} \mathbf{a}_j c_j$, where $c_j \geq 0$, and (ii) when there is no such column. In case (i), at least one c_j is nonzero, and thanks to the inductive hypothesis for the matrix $A_{(i)}$, i.e., the $m \times (n-1)$ matrix obtained from A by deleting i-th column, there exists a $y^* \in \mathbb{R}^m$ such that $y^{*\prime} \cdot A > 0$. We can then put $x^* = 0$. In case (ii), we form the subspace W spanned by the n-th column of A, $\mathbf{a}_n$, and its negative $-\mathbf{a}_n$. If there is a nonzero nonnegative solution x_r^* to $A_r^{\perp} \cdot x_r = 0$, then $A_r x_r^* = 0 + w$ for some $w \in W$. Since we are in case (ii), we have $w = -\mathbf{a}_n c_n$ for some $c_n \geq 0$. (This is because, if $w = \mathbf{a}_n c_n$ for $c_n > 0$, this should have been in case (i).). Then, $x^* \equiv (x_r^*, c_n)'$ is a nonzero nonnegative solution to $Ax = 0$, a contradiction. Thus, there exists no nonzero nonnegative solution to the equation $A_r^{\perp} \cdot x_r = 0$, where $r = n - 1$. By the inductive hypothesis we can find $y_{W^{\perp}}$ such that $y'_{W^{\perp}} \cdot A_r > 0$ and $y'_{W^{\perp}} \cdot \mathbf{a}_n = 0$. By considering $y^* \equiv c \cdot y_{W^{\perp}} + \mathbf{a}_n$, we have $y^{*\prime} \cdot A > 0$ for a sufficiently large positive number c. □

It is easy enough to obtain the Farkas-Minkowski theorem from Tucker's key theorem by a simple algebraic proof. See, for example, Nikaido (1968: Corollary 1, p.38).

5. CONCLUDING REMARKS

Let us summarize our method of analytic proof so that it might become useful for easier proofs of other annoying basic theorems, which have some equivalent propositions. Let us call two equivalent Propositions, A (here, the Minkowski-Farkas theorem) and B (the closedness of a finitely generated cone). Suppose these two propositions involve the dimension, or the index n. When it is easy to prove B from A for any index n, and it is possible to prove A with the index n from B with its index $n-1$, then we can work out our tandem method, provided A and B are valid when $n=1$. A contrivance may be required to prove A with n from B with $n-1$, because here is a kind of 'fault'. Our analytic proof is admittedly not very elementary. Proposition 2 is, however, a very basic tool in economics, and it is stimulating to economics students to show the tool is also useful in proving a fundamental mathematical theorem. It should be noted that our analytic proof cannot be extended to a nonlinear generalization of the Farkas-Minkowski theorem simply because Proposition 1 in Section 3 is an easy one only when a given system is linear.

The second remark is concerned the comparison between our second proof with the Gale-Tucker-Bartl algebraic proof. Bartl's adaptation is remarkable in the fact that it does not need the inner-product for vector spaces. Thus, Bartl had to consider the 'orthogonalization' of given linear functionals against only one special vector found by the inductive hypothesis. This has made Gale's and Bartl's proofs a little difficult to grasp geometrically. Our algebraic proof, on the other hand, used the orthogonalization of column vectors against a subspace, and the resulting decomposition of a given vector space to a direct sum of two subspaces. Though longer than Bartl's, our proof may be easier to understand geometrically what we are doing through it.

The final remark is about Stiemke's contributions to systems of linear equations and inequalities. Stiemke (1915) includes in fact four theorems, and his third and fourth have rarely been mentioned in the literature. Tucker (1956) touches upon the first two theorems only, and Good (1959) does not even cite Stiemke's paper. It seems to the present authors, however, that Stiemke's theorems II (equivalent to Gordan's theorem) and III together are really a more precise form of Tucker's key theorem. That is, in Tucker's key theorem, we can assert that $x_i^* \cdot (A'y^*)_i = 0$ for all i. This follows from a seemingly natural interpretation of Stiemke's regular variables as those positive entries in $\overline{x}$ of Sections 3 and 4, and his singular variables as those correspond to the zero entries in $\overline{x}$. Though Stiemke did not present his proofs for theorems III and IV, we may propose to rename Tucker's key theorem the 'Stiemke-Tucker key theorem'.

APPENDIX

We prove the following proposition which is more general than Proposition 2 in the text.
Proposition. Let $f(x)$ be a function for $x \in \mathbb{R}^n$. If the partial derivative exists and continuous at each x, then the assertions of Proposition 2 hold.
Proof. We use mathematical induction on n. In the case $n=1$, we know $r=1$, and we

can put

$$\lambda = \frac{\partial f(x)}{\partial x_1}\bigg|_{x=x^*}.$$

We assume the proposition is valid up to the case $(n-1)$ with $1 \leq r \leq (n-1)$, and prove the case n. Let us suppose without the loss of generality that $x_1^* > 0$. When $r < n$, we fix $x_n = x_n^*$, and apply the proposition for the case $(n-1)$, obtaining the desired result for the first $(n-1)$ variables. For the last variable x_n, we have

$$f(x_1^*, \dots, x_n^* + t) \geq f(x_1^*, \dots, x_n^*) \text{ for } t > 0,$$

because $f(x^*)$ is the minimum value. From this it follows

$$\frac{f(x_1^*, \dots, x_n^* + t) - f(x_1^*, \dots, x_n^*)}{t} \geq 0.$$

By letting t approach to zero, we get

$$\frac{\partial f(x)}{\partial x_n}\bigg|_{x=x^*} \geq 0.$$

When $x_n^* > 0$, we can take a small negative value for t in the top inequality above so that $x_n^* + t \geq 0$, thus getting

$$f(x_1^*, \dots, x_n^* + t) \geq f(x_1^*, \dots, x_n^*) \text{ for } t < 0.$$

From this,

$$\frac{f(x_1^*, \dots, x_n^* - (-t))) - (f(x_1^*, \dots, x_n^*)}{-t} \geq 0.$$

Again by letting t approach to zero, we get

$$-\frac{\partial f(x)}{\partial x_n}\bigg|_{x=x^*} \geq 0. \text{ That is, } \frac{\partial f(x)}{\partial x_n}\bigg|_{x=x^*} \leq 0.$$

Therefore, when $x_n^* > 0$, we should have

$$\frac{\partial f(x)}{\partial x_n}\bigg|_{x=x^*} = 0.$$

Now let us proceed to the case $r = n$. When we fix $x_n = x_n^*$, and apply the proposition for the case $(n-1)$ to the first $(n-1)$ variables after suitable transformations of the variables such that $y_j \equiv \alpha x_j$ for $1 \leq j \leq (n-1)$ with $\alpha > 0$ in order to have $\sum_{j=1}^{(n-1)} y_j = 1$, we get the desired results for those variables, and we know

$$\frac{\partial f(x)}{\partial x_1}\bigg|_{x=x^*} = \lambda$$

for a certain number λ because $x_1^* > 0$. Since $f(x^*)$ is the minimum value, we have

$$f(x_1^* - t, \ldots, x_j^*, \ldots, x_n^* + t) \geq f(x_1^*, \ldots, x_j^*, \ldots, x_n^*) \text{ for } x_1^* \geq t > 0.$$

From this we have

$$\frac{f(x_1^* - t, \ldots, x_n^* + t) - f(x_1^*, \ldots, x_n^* + t) + f(x_1^*, \ldots, x_n^* + t) - f(x_1^*, \ldots, x_n^*)}{t} \geq 0.$$

By taking the limit of t, we get

$$-\frac{\partial f(x)}{\partial x_1}\bigg|_{x=x^*} + \frac{\partial f(x)}{\partial x_n}_{x=x^*} \geq 0, \text{ that is, } \frac{\partial f(x)}{\partial x_n}_{x=x^*} \geq \lambda.$$

(Here, we have needed a theorem concerning the convergence of a double sequence (Apostol (1974, p.231)). For any sequence $\{t_j\}$, $t_j \to 0$, with $t_j > 0$,

$$\frac{f(x_1^* - t, \ldots, x_n^* + t) - f(x_1^*, \ldots, x_n^* + t)}{t}$$

produces a double sequence by choosing two indices for the first and n-th variables respectively. We assume the denominator t changes together with the t for the first variable. While fixing the index for the first variable, say at $\bar{t}$, we can consider the limit of the sequence created by changing t in the last variable. This sequence converges to a uniformly continuous function

$$\frac{f(x_1^* - \bar{t}, \ldots, x_n^*) - f(x_1^*, \ldots, x_n^*)}{\bar{t}}$$

on a suitably chosen compact domain including x^* in its interior. The convergence is uniform because the limit function is a uniformly continuous function. Thus the double limit coincides with the iterated limit obtained by $\bar{t} \to 0$.)

When $x_n^* > 0$, we can choose a sufficiently small negative value for t so that $x_n^* + t \geq 0$. Through a similar argument to the above, it follows

$$\frac{\partial f(x)}{\partial x_n}\bigg|_{x=x^*} \leq \lambda.$$

Hence, when $x_n^* > 0$, we should have

$$\frac{\partial f(x)}{\partial x_n}\bigg|_{x=x^*} = \lambda.$$

This completes the proof. □

We note that a quadratic function satisfies the required conditions.

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