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A Note on Consumer Surplus and the Structure of Preferences

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Abstract

We represent quasi-linear preferences by the dual measure of consumer surplus, and investigate demand and the associated multiproduct pricing. In particular, we discuss the role of substitutability "within group" and with the outside commodity, deriving a Slutsky-like decomposition of the price effect. We use our results to show that Ramsey prices are proportional to marginal costs only if preferences are fully homothetic, and that commodities with larger outside substitutability have smaller Lerner indexes.

JEL Classification: D11; D43; D61.
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1 Introduction

By making the assumption of quasi-linear preferences, consumer behaviour can be analysed focusing on the restricted number of goods included into the concave component of the utility function. The other goods, which are assumed to have constant prices, are represented by a "composite commodity" entering utility linearly. This approach is common practice in partial equilibrium applications because it simplifies the resulting demand system. In particular, at the cost of neglecting the role of available income, it makes its Jacobian a symmetric, negative definite matrix, and it allows the use of consumer surplus as an exact welfare metric: see e.g. Varian (1992: chapter 10).

Obviously, the structure of the preferences captured by the (sub-) utility function matters, and affects the demand system. In fact, it creates a number of price effects, which mix the traditional income and substitution effects with

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those due to the substitutability with respect to the outside commodity. The purpose of this note is to illustrate these effects, and some of their economic implications, which to some extent tend to be overlooked by practioners using the assumption of quasi linearity. In particular, we first represent quasi-linear preferences by means of the dual measure of consumer surplus *as a function of prices* (in fact, an example of a normalized *profit* function), and investigate the properties of direct demands. These can be interpreted as so-called "Frisch demands" (for which after a price change income is compensated in such a way to leave unchanged its marginal utility), as well as "unconditional input demands" (for which the price of output/utility is kept constant). We then discuss as the inverse demand system can be analogously examined.

To disentangle the role of the structure of preferences, we derive a Slutsky-like equation which decomposes the impact of price changes into "within-group" (compensated) substitution effects, and "size effects". The latter result from the interaction of the usual income effects, which can still be identified, and of the substitutability with the outside commodity. In particular, following the analysis by Browning (1985) of demands *à la* Frisch, we identify both a measure of "overall" outside substitutability, and a measure of "idyosincratic" substitutability which applies to single commodities. After discussing the special cases of homotheticity and additivity of the (sub-) utility function, we illustrate our results by applying them to the Ramsey pricing problem. In particular, we show that Ramsey prices will be *always* proportional to marginal costs (no matter their values) only if preferences are homothetic, and that Ramsey Lerner indexes will be smaller for commodities with larger outside substitutability.

The present work was inspired by Armstrong and Vickers (2015) (henceforth AV), who characterize a class of consumer preferences which generalises homotheticity. By using the inverse demand system, they show (among other interesting results)[1] that it is if and only if preferences belong to this class (henceforth AVP) that consumer surplus (henceforth surplus), as a function of quantities, is homothetic. And that in this case Ramsey quantities are proportional to their efficient values (and relative price-cost margins are constant). Thus, in addition to contributing to the theory of consumer demand (see e.g. Deaton and Muelbauer, 1980 and Browning, 1985), our note belongs to the large literature which has explored Ramsey pricing (see e.g. Bös, 1994).

This note is organised as follows. In section 2 we represent quasi-linear preferences by means of the dual measure of surplus, and investigate the properties of direct demand. In section 3 we use our results to study the case for Ramsey prices to be proportional to marginal costs. In section 4 we discuss a few other implications of our results for Ramsey pricing, unraveling the role of substitutability "within group" and with the outside commodity. In section 5 we discuss the "inverse" measures of outside substitutability that can be derived from the inverse demand system, and their main implication for Ramsey pricing. Finally, in section 6 we summarize the contribution by AV and represent

[1] AV also discuss the relationship between Ramsey pricing and market equilibria, and the case for decentralising the choice of relative quantities.

the AVP class in terms of our dual measure.

2 A Dual Surplus Measure

When preferences are quasi-linear they can be represented by the utility function $U = x_0 + u(\mathbf{x})$, where u is increasing and strictly concave ($u(\mathbf{0}) = 0$), and $\mathbf{x}$ is a vector of the quantities of $n \geq 2$ goods (x_0 is the amount of the outside commodity). In this note we are concern with the structure of preferences captured by the (sub-) utility function u, and with its implications for surplus and pricing. In particular, in this section we discuss a surplus measure which provides a representation of preferences dual to u.

Consider

$$\widetilde{s}(\mathbf{p}) \equiv \max_{\mathbf{x}} \{u(\mathbf{x}) - \mathbf{p}'\mathbf{x}\} \tag{1}$$

$$= u(\mathbf{x}(\mathbf{p})) - \mathbf{p}'\mathbf{x}(\mathbf{p}), \tag{2}$$

where

$$\mathbf{x}(\mathbf{p}) = \arg\max_{\mathbf{x}} \{u(\mathbf{x}) - \mathbf{p}'\mathbf{x}\} \tag{3}$$

is the direct demand system, whose Jacobian

$$D\mathbf{x}(\mathbf{p}) = \left[D^2 u(\mathbf{x}(\mathbf{p}))\right]^{-1} \tag{4}$$

is a symmetric, negative definite matrix. As it is evident from (1)-(2), the surplus measure $\widetilde{s}$ is the (normalized) "profit function" dual to the "production function" u, and $\mathbf{x}(\mathbf{p})$ has the properties of a profit-maximizing input demand system: see e.g. Lau (1978). In particular, it follows from Hotelling's Lemma that

$$\nabla \widetilde{s}(\mathbf{p}) = -\mathbf{x}(\mathbf{p}),$$

as is well known: thus $\widetilde{s}$ is decreasing and strictly convex.

Alernatively, we could define

$$\widetilde{s}(\mathbf{p}) \equiv V(\mathbf{p}, \chi(\mathbf{p})) - \chi(\mathbf{p}),$$

where

$$V(\mathbf{p}, E) = \max_{\mathbf{x}} \{u(\mathbf{x}) \text{ s.t. } \mathbf{p}'\mathbf{x} \leq E\}$$

is the indirect utility function dual to u, and the expenditure level E is adjusted to $\chi(\mathbf{p})$ so that

$$\frac{\partial V(\mathbf{p}, \chi)}{\partial E} \equiv 1.$$

It is easy to see that $\chi = \mathbf{p}'\mathbf{x}(\mathbf{p})$, and that

$$\mathbf{x}(\mathbf{p}) = -D_{\mathbf{p}} V(\mathbf{p}, \chi(\mathbf{p})) = \widehat{\mathbf{x}}(\mathbf{p}, \chi(\mathbf{p}))$$

can thus be interpreted as a so-called Frisch's demand system: see Frisch (1959) and Browning (1985). Notice that $\widehat{\mathbf{x}}(\mathbf{p}, E)$ is the Marshallian (uncompensated) demand system associated to u: i.e., $V(\mathbf{p}, E) = u(\widehat{\mathbf{x}}(\mathbf{p}, E))$.

One can also define a consistent measure of surplus as a function of quantities as

$$s(\mathbf{x}) \equiv u(\mathbf{x}) - \mathbf{p}(\mathbf{x})'\mathbf{x}, \tag{5}$$

where

$$\mathbf{p}(\mathbf{x}) = \nabla u(\mathbf{x}) = \mathbf{x}^{-1}(\mathbf{x}), \tag{6}$$

is the inverse demand system. Clearly, $s = \widetilde{s}(\mathbf{p}(\mathbf{x}))$. Notice that

$$\nabla s(\mathbf{x}) = -D\mathbf{p}(\mathbf{x})\mathbf{x}, \tag{7}$$

and

$$D\mathbf{p}(\mathbf{x}) = D^2 u(\mathbf{x}), \tag{8}$$

whose properties follow directly from those of u. But s need not to be increasing or concave, as noted by AV.

The direct and inverse demand systems can be used to classify commodities: see e.g. Bertoletti (2005) in the case of production. In particular, we might say that commodities i and j $(i \neq j)$ are *gross* q-substitutes when $\partial p_i/\partial x_j \leq 0$.[2] A sufficient condition for this is that *all* goods are *gross* p-substitutes, i.e., $\partial x_i/\partial p_j \geq 0$. Conversely, if *all* goods are gross p-complements, i.e., $\partial x_i/\partial p_j < 0$, they must also be *gross* q-complements, i.e., $\partial p_i/\partial x_j > 0$: see e.g. Takayama (1985: Theorem 4.D.3, p. 393). Thus, with more than 2 goods the "direct" and "inverse" classification need not to correspond each other.

We can explore the properties of $\mathbf{x}(\mathbf{p})$ by introducing the expenditure function $E(\mathbf{p}, u)$ dual to u.[3] Let

$$E(\mathbf{p}, u) = \min_{\mathbf{x}} \{\mathbf{p}'\mathbf{x} \text{ s.t. } u(\mathbf{x}) \geq u\},$$

and

$$\widetilde{\mathbf{x}}(\mathbf{p}, u) = \arg\min_{\mathbf{x}} \{\mathbf{p}'\mathbf{x} \text{ s.t. } u(\mathbf{x}) \geq u\}$$

be the Hicksian compensated ("conditional") demand system for which $E(\mathbf{p}, u) = \mathbf{p}'\widetilde{\mathbf{x}}(\mathbf{p}, u)$. Clearly

$$\widetilde{s}(\mathbf{p}) = \max_{u} \{u - E(\mathbf{p}, u)\},$$

where the optimal "supply function"

$$\widetilde{u}(\mathbf{p}) = \arg\max_{u} \{u - E(\mathbf{p}, u)\} = u(\mathbf{x}(\mathbf{p}))$$

[2] Notice that these conditions and the corresponding taxonomies are just *local*: throughout this note, for the sake of brevity we will avoid repeating this qualification.

[3] One can investigate the dual, inverse, demand system $\mathbf{p}(\mathbf{x})$ by making use of the so-called distance function $d(\mathbf{x}, u)$: see e.g. Deaton (1979). This analysis is sketched in the Appendix (also see Bertoletti, 2005).

satisfies

$$\frac{\partial E(\mathbf{p}, \widetilde{u})}{\partial u} = 1. \tag{9}$$

It follows that $E(\mathbf{p}, \widetilde{u}(\mathbf{p})) = \chi(\mathbf{p})$, $\widetilde{s}(\mathbf{p}) = \widetilde{u}(\mathbf{p}) - E(\mathbf{p}, \widetilde{u}(\mathbf{p}))$ and $\mathbf{x}(\mathbf{p}) = \widetilde{\mathbf{x}}(\mathbf{p}, \widetilde{u}(\mathbf{p}))$.

Differentiating (9) and using Shephard's Lemma:

$$\sum_j p_j \frac{\partial x_j(\mathbf{p})}{\partial p_i} = -\psi(\mathbf{p}) \frac{\partial \widetilde{x}_i(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial u}, \tag{10}$$

where

$$\psi(\mathbf{p}) = \left[\frac{\partial^2 E(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial u^2}\right]^{-1} > 0$$

is an "absolute" measure of the curvature of the expenditure function with respect to u, whose positivity follows from u strict concavity. Notice that ψ would change for a monotonic transformation of the utility function u, and that:

$$\frac{\partial \widetilde{x}_i(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial u} = \frac{\partial \widehat{x}_i(\mathbf{p}, \chi(\mathbf{p}))}{\partial E}.$$

From (10) we get

$$\frac{\partial \chi(\mathbf{p})}{\partial p_i} = -\psi(\mathbf{p}) \frac{\partial \widetilde{x}_i(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial u} + x_i(\mathbf{p}), \tag{11}$$

and by moving to the elasticities and summating across commodities[4]

$$\frac{\psi(\mathbf{p})}{\chi(\mathbf{p})} = 1 - \sum_i \frac{\partial \ln \chi(\mathbf{p})}{\partial \ln p_i} = 1 - \frac{\partial \ln \chi(\rho \mathbf{p})}{\partial \ln \rho} \Big|_{\rho=1}. \tag{12}$$

From (12) ψ/χ can be interpreted as a measure of the "size" properties of the preferences. In particular, it measure the overall substitutability between x_0 and $\mathbf{x}$,[5] telling us how much the overall expenditure changes for a proportional increase of all prices: it is not obvious that the expenditure χ decreases, nor that $\partial\chi/\partial p_i < 0$. Indeed, from (11) the expenditure increases if the price of an "inferior" good i (in terms of u: i.e., $\partial \widetilde{x}_i/\partial u = \partial \widehat{x}_i/\partial u < 0$) rises.

Manipulating (10) we also get:

$$\frac{\partial \ln x_i(\rho \mathbf{p})}{\partial \ln \rho} \Big|_{\rho=1} = -\phi(\mathbf{p}) \frac{\partial \ln \widetilde{x}_i(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial \ln u} = -\theta_i(\mathbf{p}), \tag{13}$$

where

$$\phi(\mathbf{p}) = \frac{\psi(\mathbf{p})}{\widetilde{u}(\mathbf{p})}$$

[4] Notice that:

$$\sum_j \frac{p_j \widetilde{x}_j(\mathbf{p}, u)}{E(\mathbf{p}, u)} \frac{\partial \ln \widetilde{x}_j(\mathbf{p}, u)}{\partial \ln u} = \frac{\partial \ln E(\mathbf{p}, u)}{\partial \ln u}.$$

[5] The larger is ψ/χ, the larger is the substitutability with the outside commodity.

is a "relative" measure of the expenditure curvature with respect to u.[6] (13) decomposes the impact of a proportional change of all prices on the consumption of commodity i (a measure of its substitutability with respect to the outside commodity) into an idyosincratic term, given by $\partial \ln \widetilde{x}_i / \partial \ln u$, and a common "size" component measured by ϕ. It is convenient to identify this kind of "outside substitutability" with a specific measure, and we will use $\theta_i(\mathbf{p})$.[7] (11) and (13) somehow suggest that $\partial \chi / \partial p_i$ will be smaller in absolute value (and possibly positive) for commodity with a smaller θ_i. Notice that:

$$\sum_i \vartheta_i(\mathbf{p})\theta_i(\mathbf{p}) = \frac{\psi(\mathbf{p})}{\chi(\mathbf{p})},$$

where $\vartheta_i = p_i x_i / \chi$ is the expenditure share of commodity i, which says that the average outside effect across commodities is just measured by ψ/χ.

We can now derive a sort of Slutsky's equation:[8]

$$\frac{\partial x_i(\mathbf{p})}{\partial p_j} = \frac{\partial \widetilde{x}_i(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial p_j} - \psi(\mathbf{p}) \frac{\partial \widetilde{x}_i(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial u} \frac{\partial \widetilde{x}_j(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial u}. \qquad (14)$$

This says that the effect of a price change can be decomposed into a compensated, "within-group" substitution effect, and a "size effect". The latter depends on the overall ouside sustitutability captured by ψ and on the traditional "income effects" (in term of the preferences represented by u). In particular, even if commodities i and j are *net* p-substitutes, i.e., $\partial \widetilde{x}_i / \partial p_j \geq 0$, if they are *both* either "normal" ($\partial \widetilde{x}_i / \partial u$, $\partial \widetilde{x}_j / \partial u > 0$) or "inferior" ($\partial \widetilde{x}_i / \partial u$, $\partial \widetilde{x}_j / \partial u < 0$) they might turn into *gross* p-complements, i.e., the cross derivative $\partial x_i / \partial p_j$ could be negative.

Moving to the elasticities we get:

$$\frac{\partial \ln x_i(\mathbf{p})}{\partial \ln p_j} = \frac{\partial \ln \widetilde{x}_i(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial \ln p_j} - \phi(\mathbf{p}) \frac{p_j x_j(\mathbf{p})}{\widetilde{u}(\mathbf{p})} \frac{\partial \ln \widetilde{x}_j(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial \ln u} \frac{\partial \ln \widetilde{x}_i(\mathbf{p}, \widetilde{u}(\mathbf{p}))}{\partial \ln u}.$$

Rewriting last expression in a perhaps more transparent form we get:

$$\frac{\partial \ln x_i(\mathbf{p})}{\partial \ln p_j} = \vartheta_j(\mathbf{p}) \left[\sigma_{ij}(\mathbf{p}, \widetilde{u}(\mathbf{p})) - \frac{\theta_i(\mathbf{p})\,\theta_j(\mathbf{p})}{\psi(\mathbf{p})/\chi(\mathbf{p})} \right], \qquad (15)$$

[6] ϕ and ψ are closely related respectively to Frisch (1959) "money flexibility":

$$\frac{\partial \ln \left\{ \frac{\partial V(\mathbf{p}, E)}{\partial E} \right\}}{\partial \ln E},$$

and Houtthaker (1960) "income flexibility":

$$\left[\frac{\partial \ln \left\{ \frac{\partial V(\mathbf{p}, E)}{\partial E} \right\}}{\partial \ln E} E \right]^{-1}.$$

[7] The larger is θ_i, the larger is the substitutability of good i with the outside commodity (notice that θ_i is negative for "inferior" commodities).

[8] The equivalent of this equation is somehow known in production theory: see for instance Bertoletti, (2005: p. 188).

where

$$\sigma_{ij}(\mathbf{p},u)=\frac{\partial\ln\widetilde{x}_i(\mathbf{p},u)}{\partial\ln p_j}\frac{E(\mathbf{p},u)}{p_j\widetilde{x}_j(\mathbf{p},u)}$$

is the so-called Allen-Uzawa elasticity of substitution between commodities i and j (AUES: see e.g. Bertoletti, 2005), a well-known measure of compensated substitutability. Accordingly, two commodities i and j ($i\neq j$) are gross p-complements if their "within-group" substitutability (measured by σ_{ij}) is smaller than a threshold determined by the product of their measures of idyosincratic outside substitutability, divided by the "scale" term ψ/χ which measures the average ouside substitutability. Notice that

$$\frac{\partial\ln x_i\left(\mathbf{p}\right)}{\partial\ln p_i}=\vartheta_i(\mathbf{p})\left[\sigma_{ii}(\mathbf{p},\widetilde{u}(\mathbf{p}))-\theta_i(\mathbf{p})^2\frac{\chi(\mathbf{p})}{\psi(\mathbf{p})}\right] \tag{16}$$

offers a decomposition of the own price effect into a substitution and a size effect which are both necessarily negative.

It matters recording the results which apply to the case of homothetic preferences, i.e., when u is homothetic. A function f is homothetic if and only if $f(\mathbf{x})=f(\mathbf{y})$ implies $f(\rho\mathbf{x})=f(\rho\mathbf{y})$ for all $\rho>0$ and $\mathbf{x}$ and $\mathbf{y}$ which belong to its domain. This definition is equivalent to the possibility of writing f as a monotonic transformation of a linear homogeneous function. A useful fact is recorded in the following:

Lemma1. *A differentiable function $f\left(\mathbf{z}\right)$ with at least one strictly nonzero derivative is homothetic if and only if $\left(\partial f/\partial z_i\right)/\left(\partial f/\partial z_j\right)$ is homogeneous of degree zero with respect to $\mathbf{z}$, $i,j=1,..,n$.*

Proof: see Lau (1969: Lemma 1, p. 379).

Using Lemma 1 one can prove that $\widetilde{s}$ is homothetic if and only if u is homothetic (and x_i/x_j is homogenous of degree zero): see Lau (1978: Theorem *II*-2, p. 153). Under homotheticity the size effects are the same across commodities (apart from the role of the expenditure shares), and are determined only by the "scale" elasticity ψ. In particular, since $\partial\ln\widehat{x}_i/\partial\ln E=1$ and $\partial\ln\widetilde{x}_i/\partial\ln u=\partial\ln E/\partial\ln u$ we get:

$$\frac{\partial\ln x_i\left(\mathbf{p}\right)}{\partial\ln p_j}=\vartheta_j(\mathbf{p})\left[\sigma_{ij}(\mathbf{p})-\frac{\psi(\mathbf{p})}{\chi(\mathbf{p})}\right],\quad\sum_j\frac{\partial\ln x_i(\mathbf{p})}{\partial\ln p_j}=-\frac{\psi(\mathbf{p})}{\chi(\mathbf{p})}. \tag{17}$$

Notice that in this case the AUES does not depend on the level of utility (but it might depend on relative prices).

It is also worth exploring a little bit more the case of additivity, i.e., when u is additive. It can be proven that $\widetilde{s}$ is additive if and only if u is additive: see Lau (1978:Theorem *II*-7, p. 158). In such a case it is known that

$$\frac{\partial\widetilde{x}_i\left(\mathbf{p},u\right)}{\partial p_j}=-\frac{\frac{\partial\widehat{x}_i(\mathbf{p},E(\mathbf{p},u))}{\partial E}\frac{\partial\widehat{x}_j(\mathbf{p},E(\mathbf{p},u))}{\partial E}}{\frac{\partial^2V(\mathbf{p},E(\mathbf{p},u))}{\partial E^2}}\frac{\partial V\left(\mathbf{p},E\left(\mathbf{p},u\right)\right)}{\partial E}$$

$i \neq j$, and

$$\frac{\partial \widetilde{x}_i(\mathbf{p},u)}{\partial p_i} = -\frac{\frac{\partial V(\mathbf{p},E(\mathbf{p},u))}{\partial E}}{p_i \frac{\partial^2 V(\mathbf{p},E(\mathbf{p},u))}{\partial E^2}} \frac{\partial \widehat{x}_i(\mathbf{p},E(\mathbf{p},u))}{\partial E}\left(1 - p_i \frac{\partial \widehat{x}_i(\mathbf{p},E(\mathbf{p},u))}{\partial E}\right):$$

on these and the following results see Deaton and Muellbauer (1980: pp. 137-41).

Since it is easy to see that $\partial^2 V(\mathbf{p}, \chi(\mathbf{p}))/\partial E^2 = -\psi(\mathbf{p})^{-1}$, we get

$$\sigma_{ij}(\mathbf{p}, \widetilde{u}(\mathbf{p})) = \frac{\theta_i(\mathbf{p})\,\theta_j(\mathbf{p})}{\psi(\mathbf{p})/\chi(\mathbf{p})} \tag{18}$$

and

$$\frac{\partial \ln x_i(p_i)}{\partial \ln p_i} = -\theta_i(\mathbf{p}). \tag{19}$$

(18) explains, in a sense, why under additivity commodities are gross p-independent: it happens in such a case that the substitution effect is equal to the relevant size effect. In particular, notice that the measures of idyosincratic substitutability are tied together in such a way that $\sigma_{ij}/\theta_i = \partial \ln \widehat{x}_j / \partial \ln E$. (19) shows that demand elasticity will be higher for those commodities which also have a larger outside substitutability. Notice, from (13), a full proportionality of the own price effect and of the "income" effect: i.e., "luxuries" (according to the sub-utility u) have larger price elasticites. This is an exact form of the so-called "Pigou's Law": see Deaton (1974) and Browning (1985). (19) also shows that additivity and concavity of u rule out the possibility that commodities are "inferior" (or, from (18), net p-complements).

To illustrate, consider the homothetic $u = [\sum_{i=1}^n x_i^\rho]^{\frac{1}{\eta}}$, where $\eta > \rho$, $1 > \rho > 0$ (if $\eta = 1$ we have the familiar additive CES case) and $\sigma_{ij} = \sigma = 1/(1-\rho) > 1$ is the elasticity of substitution. Then

$$E = u^{\frac{\eta}{\rho}} \left[\sum_j p_j^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \quad \text{and} \quad \widetilde{x}_i = p_i^{-\sigma} u^{\frac{\eta}{\rho}} \left[\sum_j p_j^{1-\sigma}\right]^{-\frac{1}{\rho}}.$$

Thus

$$\begin{aligned}
\widetilde{u} &= \left(\frac{\rho}{\eta}\right)^{\frac{\rho}{\eta-\rho}} \left[\sum_j p_j^{1-\sigma}\right]^{\frac{1-\rho}{\eta-\rho}}, \quad \chi = \left(\frac{\rho}{\eta}\right)^{\frac{\eta}{\eta-\rho}} \left[\sum_j p_j^{1-\sigma}\right]^{\frac{1-\rho}{\eta-\rho}}, \\
\widetilde{s} &= \eta^{-\frac{\eta}{\eta-\rho}} \rho^{\frac{\rho}{\eta-\rho}} (\eta-\rho) \left[\sum_j p_j^{1-\sigma}\right]^{\frac{1-\rho}{\eta-\rho}}, \quad x_i = \left(\frac{\rho}{\eta}\right)^{\frac{\eta}{\eta-\rho}} p_i^{-\sigma} \left[\sum_j p_j^{1-\sigma}\right]^{\frac{1-\eta}{\eta-\rho}}.
\end{aligned}$$

Since

$$\psi = \frac{\left(\sum_j p_j^{1-\sigma}\right)^{\frac{1-\rho}{\eta-\rho}}}{\rho^{\frac{\eta}{\rho-\eta}} \eta^{\frac{\rho}{\eta-\rho}} (\eta-\rho)},$$

it follows that

$$\phi = \frac{\rho}{\eta - \rho} = \theta_i, \quad \text{and} \quad \psi/\chi = \frac{\eta}{\eta - \rho}:$$

accordingly, the (common) substitutability with the outside commodities decreases with respect to the value of the iso-elasticity parameter η. Note that commodities are all "normal" and that they revert from net p-substitutes to gross p-complement if $\eta < 1$.

3 Proportional Ramsey Prices

After the seminal work of Ramsey (1927), who noted the proportionality of optimal *quantities* under certain conditions, as far as we know the use of proportional *prices* when departing from marginal costs in a multiproduct setting was first considered by Frisch (1939), in his criticism of the marginal pricing proposal by Hotelling (1938).[9] It was later suggested by Allais (1948), but mainly for practical purposes (according to Laffont and Tirole, 1993: p. 30). Interestingly, this pricing behaviour (sometimes called "Allais Rule") was used at the Electricité de France even when this was managed by Marcel Boiteaux, a prominent supporter of Ramsey Pricing: see Laffont and Tirole (1993: p. 201). That it corresponds to an optimal pricing rule when preferences are homothetic is a fact which is certainly well known to the experts of the field.[10] However, we have been unable to find in the literature the statement that also the reverse results holds, i.e., that Ramsey prices are proportional to marginal costs *only if* preferences are homothetic. For this reason we first use our results to prove the following:

Proposition 1. *Ramsey prices are proportional to (constant) marginal costs if and only if u is homothetic.*

Proof. Since sufficiency seems to be well known, we only prove the necessary part. Let us write $\widetilde{\pi}_i = (p_i - c_i)x_i$ for the profit made from commodity i, $i = 1, .., n$ (for the sake of simplicity we refer to constant unit costs c_i, but results generalize to the case in which the overall cost function $c(\mathbf{x})$ is linear homogeneous). Then overall profit is given by

$$\widetilde{\pi}(\mathbf{p}) = \sum_{i=1}^{n} \widetilde{\pi}_i(\mathbf{p}) = (\mathbf{p} - \mathbf{c})' \mathbf{x}(\mathbf{p}),$$

and we can write

$$\widetilde{W}(\mathbf{p}) = \widetilde{\pi}(\mathbf{p}) + \alpha \widetilde{s}(\mathbf{p})$$

for the Ramsey objective function (where $1 \geq \alpha \geq 0$ is the welfare weight of surplus). The first-order conditions for Ramsey pricing are then given by

[9] Hotelling (1939) reassessed the support to marginal pricing established long before by Dupuit (1844).

[10] AV refer to it as of a "familiar" result.

$(i = 1, .., n)$

$$\frac{\partial \widetilde{W}(\mathbf{p})}{\partial p_i} = x_i\left(\mathbf{p}\right) + \sum_{j=1}^{n}\left(p_j - c_j\right)\frac{\partial x_j\left(\mathbf{p}\right)}{\partial p_i} - \alpha x_i\left(\mathbf{p}\right) = 0,$$

which can be rewritten as

$$\sum_{j=1}^{n}\left(p_j^R - c_j\right)\frac{\partial x_j\left(\mathbf{p}^R\right)}{\partial p_i} = -\left(1-\alpha\right)x_i\left(\mathbf{p}^R\right) \tag{20}$$

or, in matrix form:

$$\left(\mathbf{p}^R - \mathbf{c}\right)' D\mathbf{x}(\mathbf{p}^R) = -\left(1-\alpha\right)\mathbf{x}\left(\mathbf{p}^R\right)'. \tag{21}$$

But then $\mathbf{p}^R = m^R(\alpha)\mathbf{c}$ for some optimal markup $m^R(\alpha) > 1$ requires, from (20), (10) and (13),

$$\theta_i(\mathbf{p}^R) = \frac{\left(1-\alpha\right)m^R(\alpha)}{m^R(\alpha) - 1}. \tag{22}$$

It follows that it must be the case that all the elasticities $\partial \ln \widetilde{x}_i / \partial \ln u$ are equal if Ramsey prices have to be proportional to marginal costs (i.e., the outside substitutability must be the same across goods). This is basically the requirement that the relevant u-isocline is locally linear, i.e., that u is ray-homothetic[11] at $\mathbf{x}\left(\mathbf{p}^R\right)$. But since this has to hold for any possible marginal cost vector (any possible $\mathbf{p}^R$), then the requirement is that u is ray-homothetic in the relevant portion of its domain (the image of $\mathbf{x}\left(\cdot\right)$), which under our assumption that u is concave is equivalent to full homotheticity: see e.g. Fare (1975). QED.

To get the intuition for this result, notice that $\widetilde{\pi}$ is indeed homothetic along a ray through the marginal cost vector when $\widetilde{s}$ is homothetic. In fact,

$$\nabla \widetilde{\pi}(\rho \mathbf{c}) = -\nabla s\left(\rho \mathbf{c}\right) + \frac{1-\rho}{\rho} D^2 s(\rho \mathbf{c})\rho \mathbf{c}$$

for $\rho > 0$, where the right-hand side of this expression is proportional to a vector function which is homogeneous of degree zero.[12] Accordingly, iso-surplus and iso-profit loci will be tangent in the price space along a ray through the marginal cost vector. But this tangency characterises Ramsey pricing, and fully homotheticity is necessary to achieve a similar result whatever the marginal cost vector is.

[11] A ray-homothetic function is homothetic along each ray in its domain, and possibly in different ways for different rays: see e.g. Fare (1975).

[12] Suppose that $f\left(\mathbf{z}\right)$ is homothetic: thus it can be written as $F\left(f^*\right)$, where F is monotonic and f^* is linear homogeneous. But then $\nabla f = F'\left(f^*\right)\nabla f^*$ (where ∇f^* is homogenous of degree zero), and

$$D^2 f = F''\left(f^*\right)\nabla f^* \nabla f^* + F'\left(f^*\right) D^2 f^*.$$

Accordingly,

$$D^2 f \mathbf{z} = f^* F''\left(f^*\right)\nabla f^*.$$

Finally, under homotheticity the optimal markup $m^R(\alpha)$ solves

$$\frac{m^R(\alpha)-1}{m^R(\alpha)} = (1-\alpha)\,\frac{\chi(\mathbf{p}^R)}{\psi(\mathbf{p}^R)}. \tag{23}$$

For instance, in our example:

$$m^R(\alpha) = \left[1+(1-\alpha)\,\frac{\rho-\eta}{\eta}\right]^{-1}.$$

4 Other Pricing Implications

Our results have a few other simple implications for multiproduct pricing when preferences are quasi-linear.[13] We noted in section 2 (see (19)) that, under additivity of u, the own price elasticities are higher for those commodities which also have larger outside substitutability. From (20), this trivially implies that their Ramsey Lerner indexes $L_i^R = \left(p_i^R - c_i\right)/p_i^R$ must be smaller:

$$L_i^R = \frac{1-\alpha}{\theta_i(\mathbf{p}^R)}. \tag{24}$$

However, note that (16) says that own price elasticities tend to be higher for product with larger outside substitutability even when u is *not* additive (also see next section). This suggests that Ramsey Lerner indexes should be relatively smaller in the case of "luxuries" and relatively larger in the case of "inferior" commodities.

AV show (see next sections) that $L_i^R < 0$ if and only if $\partial s\left(\mathbf{x}^R\right)/\partial x_i < 0$. As discussed in section 2, this requires that there exists at least one couple of gross p-complement goods, but as illustrated by our homothetic example this condition is far from sufficient. However, in the special case in which there are just *two* commodities (i.e., $n = 2$) we can easily state some meaningful necessary and sufficient conditions for one of them to have a negative Ramsey Lerner index. In fact, we know already (from (15)) that it is necessary that commodities are both normal (in terms of the preferences captured by u), otherwise they, being net p-substitutes could not revert to gross p-complement.

This understanding can be deepened by rewriting (21) as:

$$\begin{bmatrix} \frac{\partial \ln x_1(\mathbf{p}^R)}{\partial \ln p_1} & \frac{\partial \ln x_1(\mathbf{p}^R)}{\partial \ln p_2} \\ \frac{\partial \ln x_2(\mathbf{p}^R)}{\partial \ln p_1} & \frac{\partial \ln x_2(\mathbf{p}^R)}{\partial \ln p_2} \end{bmatrix} \begin{bmatrix} L_1^R \\ L_2^R \end{bmatrix} = \begin{bmatrix} \alpha-1 \\ \alpha-1 \end{bmatrix}. \tag{25}$$

Then by Cramer's rule $sign\left\{L_i^R\right\} = sign\left\{\frac{\partial \ln x_i(\mathbf{p}^R)}{\partial \ln p_j} - \frac{\partial \ln x_j(\mathbf{p}^R)}{\partial \ln p_j}\right\}$, $i \neq j$, $i,j = 1,2$. Accordingly, L_i^R will be negative if and only if

$$\frac{\partial \ln x_j\left(\mathbf{p}^R\right)}{\partial \ln p_j} > \frac{\partial \ln x_i\left(\mathbf{p}^R\right)}{\partial \ln p_j},$$

[13] A comparison to the classic analysis by Bös (1994: chapter 8) should convince the reader that our approach is indeed informative.

whose left-hand side is negative.

By using (15) and rearranging,[14] last inequality can be rewritten as:

$$\frac{\sigma_{ij}(\mathbf{p}^R, \widetilde{u}(\mathbf{p}^R))}{\theta_i(\mathbf{p}^R)} < \left[1 - \vartheta_i(\mathbf{p}^R)\right] \left[\frac{\partial \ln \widehat{x}_i\left(\mathbf{p}^R, \chi(\mathbf{p}^R)\right)}{\partial \ln E} - \frac{\partial \ln \widehat{x}_j\left(\mathbf{p}^R, \chi(\mathbf{p}^R)\right)}{\partial \ln E}\right]. \tag{26}$$

Since $\sigma_{ij} \geq 0$, this shows that $L_i^R < 0$ if and only if the substitutability "within group" of i, measured by the AUES, is sufficiently small with respect to its "outside substitutability", measured by $\theta_i > 0$. (26) also says that the relevant threshold for σ_{ij} depends negatively on the market share of commodity i, and that

$$\frac{\partial \ln \widehat{x}_i\left(\mathbf{p}^R, \chi(\mathbf{p}^R)\right)}{\partial \ln E} > \frac{\partial \ln \widehat{x}_j\left(\mathbf{p}^R, \chi(\mathbf{p}^R)\right)}{\partial \ln E} > 0$$

is in fact a necessary condition for $L_i^R < 0$. Notice that (26) can be rewritten (see footnote 4) as

$$\frac{\sigma_{ij}(\mathbf{p}^R, \widetilde{u}(\mathbf{p}^R))}{\theta_i(\mathbf{p}^R)} < \frac{\partial \ln \widehat{x}_i\left(\mathbf{p}^R, \chi(\mathbf{p}^R)\right)}{\partial \ln E} - \frac{\widetilde{u}(\mathbf{p})}{\chi(\mathbf{p}^R)},$$

which requires that $\partial \ln \widehat{x}_i / \partial \ln E > 1$: i.e., i must be a luxury good (in terms of u).

5 The Inverse Measures of Outside Substitutability

Since the overal profit is also given by:

$$\pi(\mathbf{x}) = (\mathbf{p}(\mathbf{x}) - \mathbf{c})' \mathbf{x}, \tag{27}$$

the Ramsey objective function can be written in term of quantities as:

$$\begin{aligned} W(\mathbf{x}) &= \pi(\mathbf{x}) + \alpha s(\mathbf{x}) \\ &= u(\mathbf{x}) - \mathbf{c}'\mathbf{x} - (1-\alpha) s(\mathbf{x}), \end{aligned}$$

and Ramsey prices must satisfy[15]

$$\mathbf{p}(\mathbf{x}^R) - \mathbf{c} = (1-\alpha) \nabla s(\mathbf{x}^R). \tag{28}$$

[14] Since

$$\sum_j \frac{\partial \ln \widetilde{x}_i(\mathbf{p}, u)}{\partial \ln p_j} = 0,$$

with only two commodities it must be the case that $(i \neq j)$

$$\sigma_{jj}(\mathbf{p}, u) = -\frac{p_i \widetilde{x}_i(\mathbf{p}, u)}{p_j \widetilde{x}_j(\mathbf{p}, u)} \sigma_{ij}(\mathbf{p}, u).$$

[15] Notice that

$$\pi^R = \left(\mathbf{p}(\mathbf{x}^R) - \mathbf{c}\right)' \mathbf{x}^R = -(1-\alpha)\, \mathbf{x}^{R\prime} D\mathbf{p}(\mathbf{x}^R)\mathbf{x}^R,$$

which says that the Ramsey overall profit will be generally positive, and that efficient pricing, i.e., $\alpha = 1$, requires $\mathbf{p}^R = \mathbf{c}$, as is of course well known.

By writing $r = \mathbf{p}(\mathbf{x})'\mathbf{x} = E(\mathbf{p}(\mathbf{x}), u(\mathbf{x}))$ for total revenue/expenditure, the "marginal revenue" of commodity j can be written according to the formula

$$\begin{aligned}\frac{\partial r(\mathbf{x})}{\partial x_j} &= \sum_k \frac{\partial p_j(\mathbf{x})}{\partial x_k} x_k + p_j(\mathbf{x}) \\ &= p_j(\mathbf{x})\left[1 - \mu_j(\mathbf{x})\right],\end{aligned}$$

where the "scale elasticity" (see e.g. Bertoletti, 2005)

$$\mu_j(\mathbf{x}) = -\sum_k \frac{\partial \ln p_j(\mathbf{x})}{\partial \ln x_k} = -\frac{\partial \ln p_j(\rho\mathbf{x})}{\partial \ln \rho} \mid_{\rho=1}$$

substitutes for the familiar "inverse demand elasticity" which appears in the single product case.

Notice that μ_k is given by (minus) the sum of the k row of the matrix of elasticities $\partial \ln p_i / \partial \ln x_j$. It can be given the interpretation of an idyosincratic, *inverse*[16] measure of outside substitutability for commodity k, since it says how much its price has to change to accomodate for a proportional increase of all the quantities. Also, it is "dual" to θ_k, in the sense that one can show that latter is given by the sum of the k row of the *inverse* of the matrix of elasticities $\partial \ln p_i / \partial \ln x_j$ (which is simply the matrix of the elasticities $\partial \ln x_i / \partial \ln p_j$), and that ray-homotheticity of u at $\mathbf{x}(\mathbf{p})$ is equivalent to both $\mu_i(\mathbf{x}(\mathbf{p})) = \mu$ and $\theta_i(\mathbf{p}) = \psi(\mathbf{p})/\chi(\mathbf{p})$, $i = 1, .., n$: see Bertoletti (2005). Notice that μ_k might be negative:[17] in such a case it is necessary to increase the price of commodity k to induce consumers to buy a proportionally larger consumption vector. Intuitively, p_k need to increase to compensate the substantial impact of the decrease of the prices of the gross q-complements of commodity k.[18]

Analogously, notice that

$$\begin{aligned}\frac{\partial \ln r(\rho\mathbf{x})}{\partial \ln \rho} \mid_{\rho=1} &= \frac{\sum_j \left[1 - \mu_j(\mathbf{x})\right] p_j(\mathbf{x}) x_j}{r(\mathbf{x})} \\ &= 1 - \mu(\mathbf{x}),\end{aligned}$$

where $\tau_j = p_j x_j / r$ is the revenue share of commodity j and $\mu = \sum_j \mu_j \tau_j$. Accordingly, we can interpret

$$\mu(\mathbf{x}) = 1 - \frac{\partial \ln r(\rho\mathbf{x})}{\partial \ln \rho} \mid_{\rho=1} = -\frac{\mathbf{x} D\mathbf{p}(\mathbf{x})\mathbf{x}}{r(\mathbf{x})} > 0$$

as an *inverse* measure of overall ("average") outside substitutability. Also notice that $(1-\alpha)\mu\left(\mathbf{x}^R\right) = \pi(\mathbf{x}^R)/r(\mathbf{x}^R)$ gives the Ramsey "rate of profit".

[16] The larger is μ_i, the smaller is the substitutability of good i with the outside commodity.

[17] The matrix of the elasticities $\partial \ln p_i / \partial \ln x_j$ is negative definite by (6). Then $\mu_k < 0$ says that it is *not* also a dominant diagonal matrix (see e.g. Takayama, 1985: chapter 4, section C).

[18] Notice that if all prices reduce, it would be possible for the consumer to buy more of each of the commodities and also of the outside good.

Armed with these measures we can rewrite (28) as

$$L_i^R = (1-\alpha)\mu_i(\mathbf{x}^R), \tag{29}$$

which generalises (24). This establishes the following result.

Proposition 2. *With constant marginal costs, Ramsey Lerner indexes are proportional to the (inverse) measures of outside substitutability.*

These results suggest that commodities could be usefully classifed also according to the *sign* of the corresponding scale elasticities (on the information provided by them see Bertoletti, 2005 and the references given there). For example, we might say that a commodity i is "subordinate" if $\mu_i < 0$ and "core" if $\mu_i \geq 0$.

6 Proportional Ramsey Quantities

The case for optimal quantities to be proportional to their efficient values was first established by Ramsey (1927), who proved the result in the case of either small price distortions or a quadratic utility. AV generalise that result by showing that the surplus measure s is homothetic if and only if preferences belong to the class AVP, for which it is possible to write $u^{AV} = h(\mathbf{x}) + g(q(\mathbf{x}))$ for some (strictly) concave function g, with h and q linear homogeneous ($g(\mathbf{0}) = 0$). In such a case

$$\mathbf{p}(\mathbf{x}) = \nabla h(\mathbf{x}) + g'(q(\mathbf{x}))\nabla q(\mathbf{x}), \tag{30}$$

$$s(\mathbf{x}) = g(q(\mathbf{x})) - g'(q(\mathbf{x}))q(\mathbf{x}) \tag{31}$$

and

$$\nabla s(\mathbf{x}) = -g''(q(\mathbf{x}))q(\mathbf{x})\nabla q(\mathbf{x}). \tag{32}$$

An intuition for this result comes again from Lemma 1: s need to be homothetic when u is homothetic (i.e., when we can write $u(\mathbf{x}) = g(q(\mathbf{x}))$), since its gradient is proportional to a function which is homogeneous of degree zero. But if utility has the alledged expression then s is unaffected by the component h.

But then, as discussed by AV, homotheticity of s implies the property that Ramsey quantities are proportional to efficient quantities (relative quantities x_i^R/x_j^R and relative price-cost margins $\left(p_i^R - c_i\right)/\left(p_j^R - c_j\right)$ do not depend on α). The intuition is that, when s is homothetic, π is homothetic along a ray through the efficient quantities. In fact, since for $\rho > 0$

$$\mathbf{p}(\rho\mathbf{x}(\mathbf{c})) - \mathbf{c} = \left[g'(\rho q(\mathbf{x}(\mathbf{c}))) - g'(q(\mathbf{x}(\mathbf{c})))\right]\nabla q(\mathbf{x}(\mathbf{c})),$$

then

$$\nabla\pi(\rho\mathbf{x}(\mathbf{c})) = \left[g'(\rho q(\mathbf{x}(\mathbf{c}))) - g'(q(\mathbf{x}(\mathbf{c})) + \rho g''(\rho q(\mathbf{x}(\mathbf{c})))q(\mathbf{x}(\mathbf{c})))\right]\nabla q(\mathbf{x}(\mathbf{c})).$$

Accordingly, the iso-loci of surplus and overall profit must be tangent in the quantity space along a ray through the efficient quantity vector.

To illustrate, consider the simple case in which:

$$h(\mathbf{x}) = \sum_{i=1}^{n} x_i \text{ and } g(q(\mathbf{x})) = \sum_{i=1}^{n} x_i^{\rho},$$

where $1 > \rho > 0$ and $g(z) = z^{\rho}$. Of course in this case $q = [\sum_{i=1}^{n} x_i^{\rho}]^{\frac{1}{\rho}}$ is the familiar CES aggregator and $u = \sum_{i=1}^{n}(x_i + x_i^{\rho})$ is additive (and actually a mixture of two CES functional forms: see Bertoletti *et al.*, 2007). Thus:

$$p_i(x_i) = 1 + \rho x_i^{\rho-1},\ \pi_i(x_i) = (1 + \rho x_i^{\rho-1} - c_i)x_i \text{ and } s(\mathbf{x}) = (1-\rho)\sum_{i=1}^{n} x_i^{\rho},$$

and the direct demand function of commodity i is given by $(i = 1, .., n)$:

$$x_i(p_i) = \left(\frac{p_i - 1}{\rho}\right)^{-\sigma},$$

where $\sigma = 1/(1-\rho)$ is the elasticity of substitution associated to q (p_i has to be larger than 1 to satisfy our mantained assumption that the outside commodity is bought, i.e., that $x_0 > 0$). Assuming $c_i > 1$, $i = 1, .., n$, it follows that

$$p_i^R = \frac{\sigma c_i + \alpha - 1}{\sigma - 1 + \alpha},$$

and

$$x_i^R = \left[\frac{\sigma(c_i - 1)}{(\sigma - 1 + \alpha)\rho}\right]^{-\sigma},\quad p_i^R - c_i = \frac{(1-\alpha)(c_i - 1)}{\sigma - 1 + \alpha}.$$

Let us conclude this note by representing the AVP class in our dual terms. Note first that surplus can be written:

$$\widetilde{s}^{AV}(\mathbf{p}) = \max_{q}\{g(q) - e^q(\mathbf{p}, q)\}, \tag{33}$$

where

$$e^q(\mathbf{p}, q) = \min_{\mathbf{x}}\{\mathbf{p}'\mathbf{x} - h(\mathbf{x}) \text{ s.t. } q(\mathbf{x}) \geq q\}, \tag{34}$$

when program (34) is well defined.[19] In such a case e^q is positive, monotonic increasing, and concave with respect to $\mathbf{p}$. Given the linear homogeneity of both $\mathbf{p}'\mathbf{x} - h(\mathbf{x})$ and $q(\mathbf{x})$, it must also be the case that it is separable: i.e.,

$$e^q(\mathbf{p}, q) = t(\mathbf{p})q,$$

with

$$\mathbf{x}^q(\mathbf{p}, q) = \arg\min_{\mathbf{x}}\{\mathbf{p}'\mathbf{x} - h(\mathbf{x}) \text{ s.t. } q(\mathbf{x}) \geq q\} = q\nabla t(\mathbf{p}).$$

Then program (33) has the solution

[19] *Sufficient* conditions are that both h and q are quasi-concave, and that $\mathbf{p}'\mathbf{x} - h(\mathbf{x})$ and $q(\mathbf{x})$ are monotonic increasing.

$$\widetilde{q}(\mathbf{p}) = g'^{-1}\left(t(\mathbf{p})\right), \tag{35}$$

implying that

$$\widetilde{s}^{AV}(\mathbf{p}) = g\left(\widetilde{q}(\mathbf{p})\right) - \widetilde{q}(\mathbf{p})t(\mathbf{p}), \tag{36}$$

$$-\nabla\widetilde{s}^{AV}(\mathbf{p}) = \mathbf{x}^{AV}(\mathbf{p}) = \mathbf{x}^{q}(\mathbf{p},\widetilde{q}(\mathbf{p})) = \widetilde{q}(\mathbf{p})\nabla t(\mathbf{p}). \tag{37}$$

and

$$\chi^{AV}(\mathbf{p}) = \widetilde{q}(\mathbf{p})\mathbf{p}'\nabla t(\mathbf{p}). \tag{38}$$

As discussed by AV, consumer choice can then be described as that of deciding upon the aggregate q as a function of the "price aggregator" t, while choosing the relative value of x_i/x_j for a given q. Note that

$$D^2\widetilde{s}^{AV}(\mathbf{p}) = -D\mathbf{x}^{AV}(\mathbf{p}) = -\frac{1}{g''\left(\widetilde{q}(\mathbf{p})\right)}\nabla t(\mathbf{p})\nabla t(\mathbf{p})' - \widetilde{q}(\mathbf{p})D^2 t(\mathbf{p})$$

is a positive definite matrix, so that $\widetilde{s}^{AV}$ is actually a decreasing and convex function of $\mathbf{p}$, i.e., a legitimate surplus function. t will be linear homogenous when preferences are actually homothetic: e^q is then a proper expenditure function, $\widetilde{s}^{AV}$ is homothetic too, $\mathbf{x}^{AV}$ is proportional to a homogeneous of degree zero function and $\chi^{AV} = \widetilde{q}(\mathbf{p})t(\mathbf{p})$.

To illustrate, reconsider the case in which $u = \sum_{i=1}^{n}\left(x_i + x_i^{\rho}\right)$: then[20]

$$t(\mathbf{p}) = \left[\sum_{i=1}^{n}\left(p_i - 1\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \quad \widetilde{q}(\mathbf{p}) = \rho^{\sigma}\left[\sum_{i=1}^{n}\left(p_i - 1\right)^{1-\sigma}\right]^{\frac{1}{\rho}}$$

and

$$\widetilde{s}(\mathbf{p}) = \rho^{\sigma-1}\left(1-\rho\right)\sum_{i=1}^{n}\left(p_i - 1\right)^{1-\sigma}, \quad \chi(\mathbf{p}) = \rho^{\sigma}\sum_{i=1}^{n} p_i\left(p_i - 1\right)^{-\sigma}.$$

The simple case of additivity suggests yet another case of proportional Ramsey quantities (beyond homotheticity). In fact, suppose that

$$\widetilde{s}(\mathbf{p}) = \sum_{j}\widetilde{s}_j(p_j),$$

where s_j, $j = 1, .., n$, are positive, decreasing and convex functions. Accordingly

$$x_i(p_j) = -\widetilde{s}_j'(p_j)$$

[20] Notice that from (13) we can compute:

$$\theta_i(p_i) = \frac{\sigma p_i}{p_i - 1},$$

which shows the impact of non-homotheticity on the outside substitutability measure.

and (20) becomes

$$\left(p_j^R - c_j\right) = -\left(1-\alpha\right)\frac{\widetilde{s}_j'(p_j^R)}{\widetilde{s}_j''(p_j^R)}.$$

But then the (negative) exponential case in which $\widetilde{s}_j = e^{-\beta_j p_j}$ with $\beta_j > 0$ must belong to the AVP class.

In fact we get:[21]

$$\widetilde{s}(\mathbf{p}) = \sum_j e^{-\beta_j p_j}, \quad x_j\left(p_j\right) = \beta_j e^{-\beta_j p_j}, \quad \chi\left(\mathbf{p}\right) = \sum_j p_j \beta_j e^{-\beta_j p_j}$$

and

$$p_j^R = c_j + \frac{1-\alpha}{\beta_j} \text{ and } x_i^R = \beta_j e^{1-\alpha-\beta_j c_j}$$

(assuming that $c_j + 1/\beta_j \leq \beta_j$, so that $p_j^R \leq \beta_j$). The dual utility function is of course given by $u = \sum_j \left[\ln \beta_j - \ln x_j + 1\right] x_j/\beta_j$, which gives raise to the homothetic $s = \sum_j x_j/\beta_j$.

7 Conclusions

Representing quasi-linear preferences by means of consumer surplus as a function of prices, we have investigated the properties of demands and derived a Slutsky-like decomposition of the price effects, distinguishing between substitutability "within group" and with the outside commodity. We have also derived the dual measures of substitutability for the inverse demand system, and applied our results to the associated Ramsey pricing problem. In particular, we have proved that Ramsey prices will be proportional to marginal costs only if preferences are homothetic, and that the optimal Lerner indexes are proportional to the measure of outside substitutability. Finally, we have represented the class of preferences recently introduced by AV in terms of our dual measure, and considered a few simple examples of it.

[21] Note that in this case:

$$\theta_i(p_i) = \beta_i p_i.$$

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Appendix: The Inverse Demand System

We can decompose the impact of quantity changes on inverse demands by introducing the distance function $d(\mathbf{x}, u)$: see e.g. Deaton (1979). This is implicitly given by

$$u(\frac{\mathbf{x}}{d}) \equiv u,$$

is increasing and linear homogeneous with respect to $\mathbf{x}$, and decreasing and (strictly, in our case) concave with respect to u. Notice that

$$\widetilde{\mathbf{p}}(\mathbf{x}, u) \equiv \nabla_{\mathbf{x}} d(\mathbf{x}, u)$$

is such that

$$E\left(\widetilde{\mathbf{p}}(\mathbf{x}, u), u\right) = 1$$

and $\widetilde{\mathbf{x}}\left(\widetilde{\mathbf{p}}(\mathbf{x}, u), u\right)$ is proportional to $\mathbf{x}$ (in addition, $d(\mathbf{x}, u(\mathbf{x})) = 1$ and $\widetilde{\mathbf{x}}\left(\widetilde{\mathbf{p}}(\mathbf{x}, u(\mathbf{x})), u(\mathbf{x})\right) = \mathbf{x}$).

It follows that

$$\widetilde{\mathbf{p}}(\mathbf{x}, u(\rho\mathbf{x})) \equiv \frac{\mathbf{p}(\rho\mathbf{x})}{r(\rho\mathbf{x})},$$

where $\rho > 0$. Differentiating with respect to x_j and ρ and evaluating at $\rho = 1$ we get:

$$\frac{\partial \widetilde{p}_i(\mathbf{x}, u(\mathbf{x}))}{\partial x_j} + \frac{\partial \widetilde{p}_i(\mathbf{x}, u(\mathbf{x}))}{\partial u} p_j(\mathbf{x}) = \frac{1}{r(\mathbf{x})} \left[\begin{array}{c} \frac{\partial p_i(\mathbf{x})}{\partial x_j} \\ -\widetilde{p}_i(\mathbf{x}, u(\mathbf{x})) \frac{\partial r(\mathbf{x})}{\partial x_j} \end{array} \right], \tag{39}$$

$$\frac{\partial \widetilde{p}_i(\mathbf{x}, u(\mathbf{x}))}{\partial u} r(\mathbf{x}) = \frac{1}{r(\mathbf{x})} \left[\begin{array}{c} \sum_k \frac{\partial p_i(\mathbf{x})}{\partial x_k} x_k \\ -\widetilde{p}_i(\mathbf{x}, u(\mathbf{x})) \sum_k \frac{\partial r(\mathbf{x})}{\partial x_k} x_k \end{array} \right] \tag{40}$$

We can rewrite (40) to get:[22]

$$\frac{\partial \ln \widetilde{p}_i(\mathbf{x}, u(\mathbf{x}))}{\partial \ln u} = \frac{u(\mathbf{x})}{r(\mathbf{x})} \left[\mu(\mathbf{x}) - \mu_i(\mathbf{x}) - 1\right].$$

In turn, plugging into (39) this delivers

$$\frac{\partial \ln p_i(\mathbf{x})}{\partial \ln x_j} = \frac{\partial \ln \widetilde{p}_i(\mathbf{x}, u(\mathbf{x}))}{\partial \ln x_j} + \left[\mu(\mathbf{x}) - \mu_i(\mathbf{x}) - \mu_j(\mathbf{x})\right] \tau_j(\mathbf{x}),$$

which is a sort of a Slutsky's equation (in elasticity terms) for the inverse demand system that can be used to study its substitutability properties (see Bertoletti, 2005).

[22] Notice that $r/u = \partial \ln u(\rho\mathbf{x}) / \partial \ln \rho \mid_{\rho=1}$.