ISSN: 2281-1346

Department of Economics and Management

DEM Working Paper Series

Generalizing Smooth Transition Autoregressions (updated version)

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 # 138 (05-17)

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May 2017

Generalizing smooth transition autoregressions

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FIRST VERSION: October 2013
THIS VERSION: May 2017

Abstract

We introduce a new time series model capable to parametrize the joint asymmetry in duration and length of cycles - the dynamic asymmetry - by using a particular generalization of the logistic function. The modelling strategy is discussed in detail, with particular emphasis on two asymmetry tests and relative diagnostics, whose power properties are explored via Monte Carlo experiments. Several case studies illustrate the high versatility of the new model, which is able to characterize the dynamic asymmetry in the cycle in different fields. In a rolling forecasting exercise our model beats its linear and conventional nonlinear competitors in point forecasting, while this superiority becomes less evident in density forecasting, specially when relying on robust measures. Finally, dynamic asymmetry is an important feature to take in account in uncertain environments.

Keywords: Dynamic asymmetry, Nonlinear time series, Econometric Modelling, Point forecasts, Density forecasts, Evaluating forecasts, Combining forecasts, Uncertainty.

JEL: C22, C51, C52

1 Introduction

Many of the economic and natural sciences time series show asymmetric fluctuations, see Tong (1990) and Teräsvirta et al. (2010) *inter alia*. For example, in Business Cycle literature, Sichel (1993) gives a double definition of asymmetry: the first - the *steepness* - happens when contractions in the levels are steeper than expansions (symmetry in the level axis); the second - the *deepness* - when the series undergoes at an accelerating time until a minimum after which it starts to recover with high, decreasing acceleration, until to smoothly recover the peak (symmetry in time axis). When these two definitions are combined, we call this *dynamic asymmetry*.

Smooth transition autoregressions (STAR), originated by the pioneering contribution by Bacon and Watts (1971) in Biostatistics, then developed in time series framework by Haggan and Ozaki (1981); Chan and Tong (1986) and Teräsvirta (1994), are currently one of the most simple and successful tools to model the nonlinear dynamics in applied literature. In particular, a logistic transition is commonly postulated when the series under consideration is assumed having asymmetric oscillations from its conditional mean. We argue that, being the logistic function reflectively symmetric by construction, the resulting logistic STAR does not match the theoretical definition of dynamic asymmetry. In other words, the available models for time series allow the econometrician, at the best, to answer to the question: *Does the series return to its original regime and when?* Here, our objective is to answer to another, more challenging question: *Is the rate of change (if any) in the left tail of the logistic transition different with respect of the right tail and how much?* As we will show, an appropriate solution to this methodological question, *per se* interesting for descriptive aims, improves the forecasting ability of STAR models family.

The econometric literature provides two strategies: the first, proposed by Sollis et al. (1999) (SLN1) and Lundbergh and Teräsvirta (2006) (LT) is to raise the STAR's transition function to an exponent using an idea by Nelder (1961); the second, suggested by Sollis et al. (2002) (SLN2) is to add a parameter inside the transition function in such a way to control for the asymmetry of both the tails of the transition function by simply using a Heaviside indicator.

Unfortunately, both of these solutions present some criticality: Figure 1, panel (a) clearly shows that in the SLN2 case, the transition function could be non-smooth; on the other hand, the SLN1 and LT parametrization, plotted in panel (b) conveys a smooth transition, but the effect of increasing of the asymmetry parameter could translate just in a shift effect, if not properly restricted as stated in the same article; moreover, this parametrization does not provide an immediate description of the behavior of each tail of the transition function (which is instead the beauty of SLN2). Thus, the detection and assessment of the dynamic asymmetry in a statistically well-specified time series model seem still an open issue.

In the next Section 2 we contribute to this strand of literature by applying to the autoregressive (AR) model a generalized version of the logistic transition function with two parameters governing the two tails of the logistic sigmoid and a logarithmic/exponential rescaling able to preserve the smoothness of the transition without requiring any additional restriction in the parameters. The resulting Generalized STAR (GSTAR) model encloses the symmetric STAR, so we modify the general-to-specific modeling procedure following Granger and Teräsvirta (1993) (GT); this is done in Section 3, where estimation and forecasting methods are also discussed. Two different LM-type tests for the null hypothesis that the two tails of the transition function are reflexively symmetric - a situation which is called *dynamic symmetry* for what follows - are built-up in Section 4. Section A.2 modifies three diagnostic tests originally introduced by Eitrheim and Teräsvirta (1996) (ET). Section 5 provides a simulation study according to which the SLT-type test seems less restrictive than the Score test. Six different case studies on U.S. industrial production and unemployment rate, International Sunspot Number, Canadian Lynx data and the Entry and Exit of US firms from the market are illustrated in Section 6, jointly with a rolling forecasting exercise where both point and density forecasting evaluation are investigated: in many of these examples, the dynamic asymmetry is found to be a non negligible feature to deal with for forecasting aims. Finally, Section 7 discusses the relevance of such a result and concludes.

2 The Model

Definition 1. Let be y_t a realization of a time series observed at $t = 1-p, 1-(p-1), \dots, -1, 0, 1, \dots, T-1, T$. Then the univariate process $\{y_t\}_t^T$ follows a GSTAR(p) model if

$$y_t = \boldsymbol{\phi}'\mathbf{z}_t + \boldsymbol{\theta}'\mathbf{z}_t G(\boldsymbol{\xi}) + \epsilon_t, \quad \epsilon_t \sim I.I.D.(0, \sigma^2), \tag{1}$$

where the $T \times 1$ vector y_t is a dependent variable, $\mathbf{z}_t = (1, y_{t-1}, \dots, y_{t-p})'$, $\boldsymbol{\phi} = (\phi_0, \phi_1, \dots, \phi_p)'$, $\boldsymbol{\theta} = (\theta_0, \theta_1, \dots, \theta_p)'$ are autoregressive parameter vectors, $\boldsymbol{\xi} = [\boldsymbol{\gamma}, h(c_k, s_t)]$ is a vector of nonlinear parameters $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ and a function of the $K = \{1, 2\}$ location parameter(s) c_k and the transition variable $s_t = y_{t-d}$, with $d > 0$ denoting the delay.

The following condition defines the class of the transition function $G(\boldsymbol{\xi})$:

Condition 1. *(i) $G(\cdot)$ is odd, monotonically increasing, possessing a nonzero derivative of order 2s + 1 in an open interval (-a, a), for $a > 0$, $s \geq 0$.*

(ii) We have $\partial^k G(\boldsymbol{\xi})/\partial^k \boldsymbol{\xi}|_{\boldsymbol{\xi}=\mathbf{0}} \neq 0$ for k odd and $1 < k < 2s+1$

We now specifies operationally this condition trough the following

Definition 2. We define the generalized transition function $G(\cdot)$ as follows:

$$G(\boldsymbol{\xi}) \doteq G(\boldsymbol{\gamma}, h(c_k, s_t)) = \left(1 + \exp\left\{ - \prod_{k=1}^{K} h\big(c_k, s_t\big)\right\}\right)^{-1}, \tag{2}$$

$$h(c_k, s_t) \doteq \begin{cases} \gamma_1^{-1} \exp(\gamma_1 |s_t - c_k| - 1) & \text{if } \gamma_1 > 0, \\ s_t - c_k & \text{if } \gamma_1 = 0, \\ -\gamma_1^{-1} \log(1 - \gamma_1 |s_t - c_k|) & \text{if } \gamma_1 < 0, \end{cases} \tag{3}$$

for $(s_t - c_k) > 0$ (or, equivalently, $h(c, s_t) > 1/2$) and

$$h(c_k, s_t) \doteq \begin{cases} -\gamma_2^{-1} \exp(\gamma_2 |s_t - c_k| - 1) & \text{if } \gamma_2 > 0, \\ s_t - c_k & \text{if } \gamma_2 = 0, \\ \gamma_2^{-1} \log(1 - \gamma_2 |s_t - c_k|) & \text{if } \gamma_2 < 0, \end{cases} \tag{4}$$

for $(s_t - c_k) \leq 0$ (or, equivalently, $h(c_k, s_t) < 1/2$).

Remark 1. Notice that the transition function $G(\cdot, \cdot, \cdot)$ is a continuous in the vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ and in the function $h(c_k, s_t)$, which is strictly increasing in s_t and c_k.

In what follows we simplify the notation by denoting the kernel of the model corresponding to the k-esim location with $\eta_{k,t} \equiv s_t - c_k$ and by $h(\eta_{k,t})$ the associated function, so that the general form of the transition function $G(\cdot)$ is equivalent to:

$$G(\boldsymbol{\gamma}, h(\eta_{k,t})) \equiv \Bigg(1 + \exp\Bigg\{ - \prod_{k=1}^{K} \Big[h\big(\eta_{k,t}\big) I_{(\gamma_1 \leq 0, \gamma_2 \leq 0)} + h\big(\eta_{k,t}\big) I_{(\gamma_1 \leq 0, \gamma_2 > 0)} + \\ + h\big(\eta_{k,t}\big) I_{(\gamma_1 > 0, \gamma_2 \leq 0)} + h\big(\eta_{k,t}\big) I_{(\gamma_1 > 0, \gamma_2 > 0)} \Big] \Bigg\} \Bigg)^{-1}. \tag{5}$$

Equation (3) (equation (4)) models the higher (lower) tail of the probability function, so allowing for the asymmetric behavior introduced by the slope parameter γ_1 (γ_2) which controls the velocity of the transition. The case in which $h(\eta_{k,t}) = \eta_{k,t}$ implies that the function nests a one-parameter symmetric logistic STAR model with slope $\gamma_1 = \gamma_2 = \gamma$. When $\gamma_1, \gamma_2 > 0$ ($\gamma_1, \gamma_2 < 0$), $h(\eta_{k,t})$ is an exponential (logarithmic) rescaling which increases more quickly (more slowly) than a standard logistic function. Model (1) can be generalized to other distributions of exponential family. The Indicator functions in (5) stress that slope parameters are not constrained, as in the classical STAR model (whereas the positiveness of the slope parameter was an identifying condition). When $\boldsymbol{\gamma} \to +\infty$, both the models nest an indicator function $I_{(s_t > c)}$, in which case they become a (Self Exciting) Threshold Autoregression

(SETAR), see Tong (1983); on the other side, they nest a straight line around 1/2 for each s_t when $\boldsymbol{\gamma} \to -\infty$.

The Generalized Logistic is plotted in Figure 2: the resulting sigmoid is clearly consistent with the Sichel (1993) definition of dynamic asymmetry (see, e.g., the case in which $\gamma_1 = -2$ and $\gamma_2 = 4$) and maintains the global slope of the transition function unchanged with respect to the traditional LSTAR one, so that no additional identification restriction is needed with respect to the traditional STAR model.

Remark 2. The model described in this section is the time series variant of the original generalized logistic model proposed by Stukel (1988), which differs for the definition of $\mathbf{z} = (x_1, \ldots, x_N)'$, for $\{x_i\}_{i=1}^N$ being N exogenous regressors, and consequently, $\eta_t = \boldsymbol{\phi}'\mathbf{z}$.

Remark 3. The generalized transition function $G(\boldsymbol{\xi}) \neq 0$ for $\boldsymbol{\gamma} = \mathbf{0}$ by definition. This is due to the fact that our Condition 1 does not include all requirements of point (ii) of the Luukkonen et al. (1988) (LST) analogue, and namely that $G(0) = 0$. An important implication of this feature is that, in principle, GSTAR(p) is able to nest a STAR(p) model, but not a linear AR(p).

For our aims, it is convenient to make this double nesting a feasible property of the GSTAR(p) process. This is possible by the following

Condition 2. *$G(\boldsymbol{\xi})$ is such that $G(\mathbf{0}, h(\cdot)) = 0$.*

Hence, an appropriate modification of $G(\cdot)$ is postulated in the next important

Remark 4. The GLSTAR(p) model described by equations (1)–(4) nests a linear AR model for $\boldsymbol{\gamma} = \mathbf{0}$ if $h(\eta_t)$ is modified as follows:

$$h(\eta_t)^{EZC} \doteq \begin{cases} \gamma_1^{-1} \exp(\gamma_1 |\eta_t| - 1) & \text{if } \gamma_1 > 0, \\ 0 & \text{if } \gamma_1 = 0, \\ -\gamma_1^{-1} \log(1 - \gamma_1 |\eta_t|) & \text{if } \gamma_1 < 0, \end{cases} \tag{6}$$

for $\eta_t \geq 0$ ($\mu > 1/2$) and

$$h(\eta_t)^{EZC} \doteq \begin{cases} -\gamma_2^{-1}\exp(\gamma_2|\eta_t| - 1) & \text{if } \gamma_2 > 0, \\ 0 & \text{if } \gamma_2 = 0, \\ \gamma_2^{-1}\log(1 - \gamma_2|\eta_t|) & \text{if } \gamma_2 < 0, \end{cases} \tag{7}$$

for $\eta_t < 0$ ($\mu < 1/2$). The label "EZC" distinguishes this version from the the original Stukel' s generalized logistic function in equations (3) – (4) for exposition matter. This special case is necessary in order to build a test for the null of linearity against of dynamic asymmetry, see next Section 4.

Remark 5. As in the traditional STAR, the process $\{\epsilon_t\}_t^T$ is assumed to be a martingale difference sequence with respect to the history of the time series up to time t - 1, denoted as $\Omega_{t-1} = [y_{t-1}, \ldots, y_{t-p}]$, i.e., $E[\epsilon_t|\Omega_{t-1}] = 0$. This is sufficient to built up tests based on artificial regressions as demonstrated in Davidson and McKinnon (1990) and has important consequence for applied aims, in what the "All-in-One" test discussed in Section 4 and the three diagnostic tests discussed in Section A.2 can still be meaningful if the normality test reject this hypothesis. For expositional purposes, we restrict the conditional variance of the process $\{\epsilon_t\}_t^T$ to be constant, $E[\epsilon_t^2|\Omega_{t-1}] = \sigma^2$. Moreover the parameter vectors $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ are assumed to not change in time and the number of regimes is assumed to not exceed $K = 2$. However, these restriction can be relaxed[1] and tested as discussed in Section A.2.

Remark 6. As in the traditional STAR, if process is characterized by $G(\mathbf{0}, h(\eta_t)^{EZC})$, we assume $Q(z) = z^p - \phi_1 z^{p-1} - \cdots - \phi_p = 0$ has its roots inside the unit circle, since this implies that the model is stationary and ergodic under the null hypothesis of linearity. Also this assumption can be relaxed, as in Kapetanios et al. (2003); Vougas (2006).

We now discuss three relevant cases of GSTAR model.

[1]For advancements in this sense see Gonzáles-Rivera (1998); Lundbergh et al. (2003); McAleer and Medeiros (2008); Amado and Teräsvirta (2013); Silvennoinen and Teräsvirta (2013).

Example 1. If $K = 1$, the parameters $\boldsymbol{\phi} + \boldsymbol{\theta} G(\boldsymbol{\gamma}, \mathbf{c}, s_t)$ change monotonically as a function of s_t from $\boldsymbol{\phi}$ to $\boldsymbol{\phi} + \boldsymbol{\theta}$. The corresponding transition function is:

$$G(\boldsymbol{\gamma}, h(\eta_{1t})) = \Big(1 + \exp\Big\{ - \Big[h(\eta_{1,t}) I_{(\gamma_1 \leq 0, \gamma_2 \leq 0)} + h(\eta_{1,t}) I_{(\gamma_1 \leq 0, \gamma_2 > 0)} + h(\eta_{1,t}) I_{(\gamma_1 > 0, \gamma_2 \leq 0)} + \\ + h(\eta_{1,t}) I_{(\gamma_1 > 0, \gamma_2 > 0)}\Big]\Big\}\Big)^{-1}, \tag{8}$$

with $h(\eta_{1,t})$ corresponding to (3) and (4).

Example 2. When $K = 2$ and $c_1 \neq c_2 = \mathbf{c}$, the model (1) nests the following STAR model with second order Generalized Logistic (GLSTAR2) function:

$$G(\boldsymbol{\gamma}, h(\eta_t)) \doteq 1 - \exp\Big\{ - h(\eta_{2,t})\Big\}, \tag{9}$$

where:

$$h(\eta_{2,t}) \doteq \begin{cases} \gamma_1^{-1} \exp(\gamma_1 |(s_t - c_1)(s_t - c_2)| - 1) & \text{if } \gamma_1 > 0, \\ (s_t - c_1)(s_t - c_2) & \text{if } \gamma_1 = 0, \\ -\gamma_1^{-1} \log(1 - \gamma_1 |(s_t - c)(s_t - c_2)|) & \text{if } \gamma_1 < 0, \end{cases} \tag{10}$$

for $(s_t - c)^2 > 0$ (or, equivalently, $h(\eta_t) > 1/2$) and

$$h(\eta_{2,t}) \doteq \begin{cases} -\gamma_2^{-1} \exp(\gamma_2 |(s_t - c)(s_t - c_2)| - 1) & \text{if } \gamma_2 > 0, \\ (s_t - c)(s_t - c_2) & \text{if } \gamma_2 = 0, \\ \gamma_2^{-1} \log(1 - \gamma_2 |(s_t - c_1)(s_t - c_2)|^2) & \text{if } \gamma_2 < 0, \end{cases} \tag{11}$$

for $(s_t - c_1)(s_t - c_2) < 0$ (or, equivalently, $h(\eta_{2,t}) < 1/2$), whith $\eta_t \equiv \eta_t = (s_t - c_1)(s_t - c_2)$. Figure 3 shows the transition function for a set of different combinations of γ_1 for fixed γ_2 (upper panel) and viceversa (lower panel).

Example 3. A particular case of GLSTAR2 holds when $K = 2$ and $c_1 = c_2 = \mathbf{c}$, in which case the model (1) nests an exponential generalized exponential autoregressive (GESTAR) model, which is defined as in (9) – (11), apart the fact that $h(\eta_{2,t}) =$

$(s_t - c)^2$ if $\gamma_1 = 0$ for $(s_t - c)^2 > 0$ and $\gamma_2 = 0$ for $(s_t - c)^2 \leq 0$. In this case, the parameters $\boldsymbol{\phi} + \boldsymbol{\theta} G(\cdot)$ change asymmetrically at some (undefined) point where the function reaches its own minimum.

A simulated example of GLSTAR model (in both Stukel and EZC versions), jointly with its symmetric Teräsvirta (1994) counterpart, is shown in Figure 4. For each of these three models, we used two different specifications, which differ for the location parameter c. As easy seen in panel (a), the Stukel and EZC versions coincides; the associated transition functions versus time plotted in panel (b) and versus ordered η_t in panel (c) confirm this finding; on the other hand, the plot of $h(\eta_t)$ versus ordered η_t in panel (d) is more informative with respect to the effect of the different kind of asymmetry in the process: $h(\eta_t)^{EZC}$ is a 45° angle straight line, while the rescaling effect is visible in the Stukel $h(\eta_t)$ parametrization.

3 Econometric Modelling and Methods

3.1 Modelling strategy

According to GT, the investigator should always be interested in testing whether a linear AR(p) representation is adequate when building a GSTAR model. If the answer is negative, then the second step will be the selection of a nonlinear symmetric model. Then, the issue of testing for dynamic symmetry hypothesis arises as further step, when finding good specifications of STAR models becomes too difficult or whenever suggested by the economic theory. The resulting General-to-Specific modelling strategy consists in the following 7 steps:

1. Specify a linear autoregressive model.
2. Test linearity for different values of d, and if rejected, determining d in (2) or (9).
3. Choose between LSTAR, LSTAR2 or ESTAR by the Teräsvirta's rule.

4. Test the symmetry of the tails transition function according to the result in Step 3.

5. If the hypothesis of symmetry is rejected, estimate the GSTAR model with the most appropriate transition function given by step 3.

6. Evaluate the new parametrization by some diagnostic tests.

7. Use the estimated GSTAR model for forecasting aims.

In our illustrations the autoregressive order p is selected according to Bayesian Information Criterion (Schwarz, 1978), which is combined with the result with a portmanteau test for serial correlation in order to avoid a wrong rejection of symmetry hypothesis. This is due to the fact that the GSTAR model requires a lower autoregressive order with respect to its symmetric counterpart.

For what concerns the Step 4, the dynamic symmetry hypothesis is tested by two different LM-type tests. In the first test the series is assumed to follow a STAR model, so that testing for symmetry is a second step with respect to testing for linearity. Hence, we will refer to this test as "Two-Step" (TS) test. In the second test we do not assume any prior of the nonlinearity of the series, so that it enclose all steps from 2) to 5) of the General-to-Specific modelling strategy above mentioned; hence, the use of the label "All-in-one" (AIO) to distinguish it from the different null hypothesis of TS test. The choice of what test to use depends on the needs of the investigator[2]. Our experience suggests to perform the AIO test should be used if the investigator wants to be conservative against evidence of asymmetric dynamics, while the TS test tends to not reject the null unless extreme situations (see Section 6 for details). Both the tests will be discussed in the next Section 4.

The choice of the delay parameter d and the choice of the transition function can be done with the same procedure adopted in Tsay (1989) and Teräsvirta (1994).

[2]Our simulation study shows that the two tests behave differently in terms of empirical power. See Section 5 for details.

3.2 Estimation

Following Leybourne et al. (1998), estimation is done by concentrating the Sum of Square Residuals function with respect to $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$, that is minimizing:

$$SSR = \sum_{t=1}^{T} \left(y_t - \hat{\boldsymbol{\psi}}' \mathbf{x_t'} \right)^2, \tag{12}$$

where:

$$\hat{\boldsymbol{\psi}} = [\hat{\boldsymbol{\phi}}, \hat{\boldsymbol{\theta}}] = \left(\sum_{t=1}^{T} \mathbf{x_t'}(\boldsymbol{\gamma}, \mathbf{c}) \mathbf{x_t}(\boldsymbol{\gamma}, \mathbf{c}) \right)^{-1} \left(\sum_{t=1}^{T} \mathbf{x_t'}(\boldsymbol{\gamma}, \mathbf{c}) y_t \right), \tag{13}$$

and

$$\mathbf{x_t}(\hat{\boldsymbol{\gamma}}, \hat{\mathbf{c}}) = \left[\mathbf{z}, \mathbf{z_t'} G(\hat{\boldsymbol{\gamma}}, h(\hat{\mathbf{c}}, s_t) \right]. \tag{14}$$

This is possible because if $\boldsymbol{\gamma}$ and $\mathbf{c}$ are known and fixed, the GSTAR model is linear in $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$, which can be easy computed via Conditional OLS (COLS). In a such a way, the nonlinear least square minimization problem, otherwise necessary, more demanding in terms of parameters to estimate and not available in closed-form, reduces to a minimization on three (four) parameters, and is solved via a grid search over γ_1, γ_2, c (c_1, c_2 in case of GLSTAR2).

In our illustrations, both γ_1 and γ_2 are generally chosen between a minimum value of -10 and a maximum of 10 with rate 0.5 in the first three examples (-150 and 150 with rate 15 in the fourth one); the grid for parameter c_1 (c_2) is the set of values computed between the 10^{th} and 90^{th} percentile of s_t with rate computed as the difference of the two and divided for an arbitrarily high number (here, 200). Anyway, this is a only gross rule of thumb and does not pretend to constitute a general indication for application to other data.

Table 1 gives an overview of the statistical accuracy of the COLS estimator in a simulation over 1000 draws. In particular, we look at the expectations of nonlinear parameters γ_1 and γ_2 given autoregressive parameters $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ and location parameter c and delay s_t. Four couples of slope parameters are choosen to understand the behavior of the estimator in case of moderate dynamic asymmetry ($\gamma_1 = 4, \gamma_2 = -1$), accentuated dynamic asymetry ($\gamma_1 = 4, \gamma_2 = -1$) no asymmetry with moderate nonlinearity ($\gamma_1 = \gamma_2 = 2$) and no asymmetry with pronounced nonlinearity ($\gamma_1 = \gamma_2 = 20$). The evidence suggests that COLS method provides quite precise estimates also when series are considerably short. Namely, in the moderate dynamic asymmetric case (first couple of columns), the estimator of γ_1 (γ_2) is under-estimated of $5,5\%$ ($12,5\%$) for T=25; then this bias reduce up to arrive, at T=300, to under-estimate its expected value of less than 1% for γ_1 (over-estimate of $4,5\%$ for γ_2). This under-evaluation of the slopes in small samples is quite more pronounced, in magnitudo, when considering high expected slope parameters as the second case in second couple of columns suggests (γ_1 is underestimated of 12,5% and γ_2 of 17,5%), while they arrive at almost their true expectation when T=300. Dynamic symmetric models, instead, are slightly more imprecise: as shown in third (fourth) couple of columns, they under-estimate γ of 18,5% (5%) when T=25 and, at T=300, they continue to under(over) of 6% (12%) for moderate (pronounced) nonlinearity.

3.3 Forecasting

The literature on point forecasting and on evaluation of individual density forecasts under linear models is nowadays established, see Timmermann (2006) and Corradi and Swanson (2006). When the model is nonlinear, the one-step forecast is immediately available if knowing the nonlinear function in what, by least-square criterion, $E(\epsilon_{t+1}|I_t) = 0$, $I_t = y_{t-i}, i \geq 1$ in (1). The multi-step ahead forecast is not available in closed form and requires numerical integration. Hence at t+1, we generate, $1, \ldots, m, \ldots, M$ draws, for example, from model (1) – (7) conditionally on the estimated nonlinear parameters $\boldsymbol{\xi}$ and obtain the forecast $y_{t+1} \sim f(y_{t+1} + \epsilon_{t+1}^{(m)}; \boldsymbol{\xi}|I_t)$, which is called skeleton forecast of y_t; in turn, this is collected to draw, at $t+2$, the forecast $y_{t+2} \sim f(y_{t+2} + \epsilon_{t+2}^{(m)}; \boldsymbol{\xi}|I_t, y_{t+1}^{(m)})$, and so on until, at $t+h$,the forecast

$y_{t+h|t} = f(y_{t+h} + \epsilon_{t+h}|I_t, y_{t+1}^{(m)}, \ldots, y_{t+h-1}^{(m)})$ is obtained and then evaluated as:

$$y_{t+h}^{MC} = \frac{1}{M}\sum_{m=1}^{M} y_{t+h|t}^{(m)}. \tag{15}$$

The Monte-Carlo approach requires to make assumptions on the distribution of random numbers ϵ_t. As we will see in Section 6, this has severe implications in our illustrations, in particular when density forecasting is required. This problem can be avoided by block-bootstrapping the original sample series. Namely, the series is divided in blocks of magnitude $b > 1$, which then are sampled with replacement and this for every possible contiguous element in the original sample; thus the sampled blocks are attached obtaining the new bootstrap series $(\tilde{y}_t^{(1)}, \ldots, \tilde{y}_t^{(i)}, \ldots, \tilde{y}_t^{(B)})$ from model (1)–(7); finally, we sequentially compute the M_B^b forecasts for $\tilde{y}_{t+1}, \tilde{y}_{t+2}, \ldots, \tilde{y}_{t+h}$ as before described up to arrive at the skeleton $\tilde{y}_{t+h} = g(\tilde{y}_{t+h} + \tilde{\epsilon}_{t+h}^{(i)}|I_{t+h-1})$ and evaluating:

$$\tilde{y}_{t+h}^{B} = \frac{1}{M_B^b}\sum_{B=1}^{M_B^b} \tilde{y}_{t+h|t}^{(B)} \tag{16}$$

In our application we adopt a moving block-bootstrap algorithm with $b = 10$ and $B = 10,000$ draws. This allows to avoid to make assumptions on the distribution of estimated residuals, and thus to have a forecast robust to model parameter uncertainty, see Efron and Tibshirani (1993), CH. 8.

The out-of-sample predictive properties of the estimated models are investigated via rolling forecast experiment, according to which the series y_t is divided in a "pre-forecast" period (going from time $\{1 \ldots t\}$) from which the model is estimated and the h-step-ahead forecast are computed and compared with the "test" period, going from time $\{T^s \ldots T\}$ where $T^s = t + h$; this allows to measure $T - T^s - h + 1$ out-of-sample forecasts. Let denote the corresponding realization of the series as y_t, y_T^s and y_T, as well as the corresponding density forecasts as f_t, f_T^s and f_T. Since our interest lies in short-run forecasting we consider $h = \{1, 3, 6, 12\}$. The point predictive performances of the model j are investigated by four different measures: the mean forecast error (MFE), the symmetric mean absolute percentage error (sMAPE), the median relative absolute error (mRAE) and the root mean square forecast error (RMSFE)[3]:

$$MFE_{j,h} = \frac{1}{T-h-T^s+1}\sum_{t=T^s}^{T-h}\left(y_{t+h} - \hat{y}^j_{t+h|t}\right) \tag{17}$$

$$sMAPE_{j,h} = \frac{100|y_{t+h} - \hat{y}^j_{t+h}|}{0.5(y_{t+h} - \hat{y}^j_{t+h|t})} \tag{18}$$

$$mRAE_{j,h} = \frac{|y_{t+h} - \hat{y}^j_{t+h}|}{|y_{t+h} - \hat{y}^{(1)}_{t+h}|}, \quad \text{with (1) indexing the benchmark model;} \tag{19}$$

$$RMSFE_{j,h} = \frac{1}{T-h-T^s+1}\sum_{t=T^s}^{T-h}\left(y_{t+h} - \hat{y}^j_{t+h|t}\right)^2 \tag{20}$$

Differently, literature on aggregation of density forecasts is instead in a development phase, and focuses on the so called *scoring rules* (or *opinion pools*), peculiar functions enabling the forecaster to properly aggregate the set of conditional predictive density as well as more common measures as Mean Square Forecast Error *et similia* do for point forecasts. Despite their dated origins in statistics, as documented by Gneiting and Raftery (2007), scoring rules are becoming increasingly applied by contemporaneous econometric literature only recently; see, *inter alia*, Geweke and Amisano (2011). In a similar fashion, concerning about density forecasting, four different scoring rules are used for aggregate the $T - T^s - h + 1$ predictive densities produced by the same forecasting exercise[4]:

- the logarithmic score (LogS):

$$LogS_{j,h} = \frac{1}{T-h-T^s+1}\sum_{t=T^s}^{T-h} log \hat{f}^j_{t+h|t} \tag{21}$$

[3] In particular, sMAPE and mRAE are recommended when the series is known to present volatility effects or skewness, two features often associated to nonlinearity; see the discussion in Tashman (2000).

[4] The scoring rules here considered are just a fraction of the many nowadays available. The choice of the scores has been done for easy of treatment and does not imply any preference.

corresponding to a Kullback-Liebler distance from the true density; models with higher LogS are preferred.

- The quadratic score, somehow the equivalent of the MSFE in point forecasting, is defined as:

$$QSR_{j,h} = \frac{1}{T-h-T^s+1} \sum_{t=T^s}^{T-h} \sum_{k=i}^{K} (f^j_{t+h|t} - d_{kt})^2 \tag{22}$$

where $d_{kt} = 1$ if $k = t$ and 0 otherwise; models with lower QSR are preferred.

- The (aggregate) continuous-ranked probability (CRPS) score, equivalent to the sMAPE, is defined as:

$$CRPS_{j,h} = \frac{1}{T-h-T^s+1} \sum_{t=T^s}^{T-h} \left(|f_{t+h} - \hat{f}^j_{t+h|t}| - 0.5|f_{t+h} - f'_{t+h}| \right), \tag{23}$$

where f and f' are independent random draws from the predictive density and $f_{t+h|t}$ the observed value; models with lower CRPS are preferred.

- Finally, the quantile score (qS) can be obtained if $f^j_{t+h|t}$ is replaced in (21) by a predictive α-level quantile $q^\alpha_{t+h|t}$ (and the logarithmic function removed); this score is used in risk analysis because provide information about deviations from the true tail of the distribution.

Once the forecasts from different models, it is interesting to ask if our model performs better than its linear and symetric competitor(s). This is done by using the Diebold and Mariano (1995)[5], the Giacomini and White (2006) and Amisano and Giacomini (2007) tests for equal predictive ability in order to compare couples of forecasts. Under the null hypothesis, there is no evidence

[5]Since the AR(p) and STAR(p) models are nested in GSTAR(p) specification here proposed, the inference of this test is severely biased, as proved in West (1996) and other equivalent tests as the one by Clark and McCracken (2001) should be employed. Nevertheless, we will still use the classical Diebold-Mariano for a matter of preliminary check. As we will see in our illustrations, the resulting p-values are often counterintuitive. In any case, the p-values of Clark-McCraken test do not change our conclusions and thus they are omitted for a matter of space.

of superiority of the GSTAR(p) - model on its linear or symmetric equivalent.

4 Testing for Dynamic Symmetry

In this section we discuss two LM-type test for the null of dynamic symmetry according to the General-to-Specific strategy stated in the previous section 3. The TS test, illustrated in Subsection 4.1, is a classical Score test on the two slope parameters and is a simple adaptation for time series of the original Stukel' s parametrization. On the other side, the AIO test, derived in Subsection 4.2, is modified version of the Taylor-expansion-based test by LST, where the $G(\boldsymbol{\gamma}, h(\eta_t^{EZC}))$ is linearized by third-order Taylor expansion in $G(\cdot)$; this leads to an augmented artificial model which in turn can be investigated by a classical χ^2 or F-test. This is due to the fact that the Information matrix is the same as in GT[6].

4.1 "Two-Step" Test

Consider the general formulation (1)-(2). Then, the null hypothesis of no logarithmic (exponential) deviations from the logistic transition in systems (1)-(8) or (1)-(11) can be tested by setting the following hypotheses testing system:

$$H_{0i} : (\gamma_1, \gamma_2) = (0, 0) \;\text{ vs }\; H_{1i} : (\gamma_1, \gamma_2) \neq (0, 0), \;\; i = 1, 2, 3 \;\; , \tag{24}$$

with subscript i indicating the type of underlining transition function, namely $i = 1$ for generalized logistic (eq. (5)), $i = 2$ for generalized second order logistic (eq. (9)) and $i = 3$ for the generalized exponential one.

This hypothesis system requires a simple score test. Let denote by $\Xi = [\boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{\gamma}, c]$ the hyper-parameter vector of the model, so that the log-likelihood function of the T observations can be denoted by $\mathcal{L}_t(\mathbf{z_t}, \Xi)$ and the score vector by $\mathbf{q_t}(\mathbf{z_t}, \Xi) =$

[6]See GT, pp. 64-5, adjust the notation for an autoregressive framework and notice that we only modify the definition of nonlinear part $f_t = f(\boldsymbol{w_t}; \boldsymbol{\psi})$, which does not vary the general result.

$\sum_t \mathbf{q}(\mathbf{z_t}, \Xi) = \partial\mathcal{L}_t(\mathbf{z_t}, \Xi)/\partial\Xi$ evaluated at $(\boldsymbol{\theta_0}, \boldsymbol{\phi_0}, \mathbf{0}, c_0)$. Then, standard results lead to the following log-likelihood function:

$$\begin{aligned}\mathcal{L}_t(\mathbf{z_t}, \Xi) &= const + \frac{T}{2}\ln\sigma^2 - \frac{1}{2}\sigma^2\sum_t(y_t - \boldsymbol{\phi}'\mathbf{z_t} - \boldsymbol{\theta}'\mathbf{z_t}G)^2 \\ &= const + \frac{T}{2}\ln\sigma^2 - \frac{1}{2}\sigma^2\sum_t u_t^2(\Xi),\end{aligned} \tag{25}$$

with *const* denoting a constant and u_t the model's residual, and to the score:

$$\mathbf{q_t}(\mathbf{z_t}, \Xi) = \sum_t \mathbf{q}(\mathbf{z_t}, \Xi) = \frac{\partial\mathcal{L}_t(\mathbf{z_t}, \Xi)}{\partial\Xi} = \frac{1}{\sigma^2}\sum_t u_t(\Xi)\mathbf{k_t} \ , \tag{26}$$

where

$$\mathbf{k_t} = \frac{\partial u_t(\Xi)}{\boldsymbol{\partial\Xi}} = (\mathbf{z_t}, \mathbf{z_t}G, \boldsymbol{\theta}'\mathbf{z_t}G_{\gamma_1}, \boldsymbol{\theta}'\mathbf{z_t}G_{\gamma_2}, \boldsymbol{\theta}'\mathbf{z_t}G_{t,c}), \tag{27}$$

with $G_{t,\gamma_1} = \partial G/\partial\gamma_1$, $G_{t,\gamma_2} = \partial G/\partial\gamma_2$, and $G_c = \partial G/\partial\gamma_c$ being defined in Appendix A.1.

Under H_0, the test statistic is:

$$S_1(\Xi)^{LM} = \frac{1}{\hat{\sigma}^2}\hat{\mathbf{u}}'\mathbf{H}(\mathbf{H}'\mathbf{H} - \mathbf{H}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{H})^{-1}\mathbf{H}'\hat{\mathbf{u}}, \tag{28}$$

where $\hat{\mathbf{u}} = [\hat{u}_1, \ldots \hat{u}_T]'$, $\mathbf{Z} = (\mathbf{z}_1', \ldots, \mathbf{z}_T')'$, $\mathbf{H} = [(\mathbf{h_1^0})', \ldots, (\mathbf{h_T^0})']'$, with $(\mathbf{h_t^0})' = \mathbf{k_t^G}$, $\mathbf{k_t^G}$ denoting the sub-vector $[\boldsymbol{\theta}'\mathbf{z_t}G_{\gamma 1}, \boldsymbol{\theta}'\mathbf{z_t}G_{\gamma 2}, \boldsymbol{\theta}'\mathbf{z_t}G_c]'$ and $n = dim(\mathbf{k_t^G})$. Under H_0, statistic S_1 is asymptotically distributed as a χ^2_n. Just minor modifications are needed in notation of $\mathbf{k_t}$ and $\mathbf{q_t}$ in case of GLSTAR2 model due to an additional c parameter with respect to the GLSTAR.

4.2 "All-in-One" Test

Consider (2) with $G(\boldsymbol{\gamma}, h(\eta_t^{EZC}))|_{\boldsymbol{\gamma}=\mathbf{0}}$ and define $\boldsymbol{\tau} = (\boldsymbol{\tau_1}, \boldsymbol{\tau_2})'$, where $\boldsymbol{\tau_1} = (\phi_0, \boldsymbol{\phi}')'$, $\boldsymbol{\tau_2} = \boldsymbol{\gamma}$. Let $\hat{\boldsymbol{\tau}}_\mathbf{1}$ the LS estimator of $\boldsymbol{\tau_1}$ under $H_0 : \boldsymbol{\gamma} = \mathbf{0}$, $\hat{\boldsymbol{\tau}} = (\boldsymbol{\tau}_\mathbf{1}', \mathbf{0}')'$. Moreover, let $\mathbf{z_t}(\boldsymbol{\tau}) = \frac{\partial\epsilon_t}{\partial\boldsymbol{\tau}}$ and $\hat{\mathbf{z}}_\mathbf{t} = \mathbf{z_t}(\hat{\boldsymbol{\tau}}) = (\hat{\mathbf{z}}_{\mathbf{1,t}}, \hat{\mathbf{z}}_{\mathbf{2,t}})$, where the partition conforms to that of

$\boldsymbol{\tau}$. Then the general form of LM statistic is:

$$S_2(\Xi)^{LM} = \frac{1}{\hat{\sigma}^2}\hat{\mathbf{u}}'\hat{\mathbf{Z}}_\mathbf{2}(\hat{\mathbf{Z}}_\mathbf{2}'\hat{\mathbf{Z}}_\mathbf{2} - \hat{\mathbf{Z}}_\mathbf{2}'\hat{\mathbf{Z}}_\mathbf{1}(\hat{\mathbf{Z}}_\mathbf{1}'\hat{\mathbf{Z}}_\mathbf{1})^{-1}\hat{\mathbf{Z}}_\mathbf{1}'\hat{\mathbf{Z}}_\mathbf{2})^{-1}\hat{\mathbf{Z}}_\mathbf{2}'\hat{\mathbf{u}}, \tag{29}$$

where $\hat{\mathbf{u}}$ is previously defined, $\hat{\sigma}^2 = \frac{1}{T}\sum_1^T \hat{u}_t^2$ and $\hat{u}_t = y_t - \hat{\boldsymbol{\tau}}_\mathbf{1}'\mathbf{z_t}$, $\hat{\mathbf{Z}}_\mathbf{i} = (\hat{\mathbf{z}}_\mathbf{i1}, \ldots, \hat{\mathbf{z}}_\mathbf{it}, \ldots, \hat{\mathbf{z}}_\mathbf{iT})'$, $i = \{1, 2\}$, $t = 1, \ldots, T$. When the model is an GLSTAR, $\hat{\mathbf{z}}_\mathbf{1,t} = -\mathbf{z_t} = -(1, y_{t-1}, \ldots, y_{t-p})'$ while $\hat{\mathbf{z}}_\mathbf{2t} \equiv \frac{\partial^2 u_t}{\partial\gamma\partial\gamma'}\big|_{\boldsymbol{\gamma}=\mathbf{0}} = -\frac{1}{2}\big\{\theta_{20}[y_t(y_{t-d})] - cy_t\boldsymbol{\theta}'\mathbf{z_t} + \boldsymbol{\theta}_\mathbf{2}'\mathbf{z_t}y_t y_{t-d}\big\}$, where d is the delay parameter. The change in the definition of $\mathbf{z_{2t}}$ is not significant in terms of LM stastistic build-up. This implies that no change of treatment with respect to the original parametrization is needed. In particular, in order to circumvent the Davies (1977)' s problem of unidentification of nuisance parameters θ_0 and $\bar{\boldsymbol{\theta}} = [\theta_1, \ldots, \theta_p]'$ under the null hypothesis, the same LST approach can be used. The linearized GLSTAR model

$$\begin{aligned} y_t = \phi'\mathbf{z_t} + \theta'\mathbf{z_t}T_3\Big[&h\big(\eta_{k,t}\big)I_{(\gamma_1\leq 0,\gamma_2\leq 0)} + h\big(\eta_{k,t}\big)I_{(\gamma_1\leq 0,\gamma_2>0)} + h\big(\eta_{k,t}\big)I_{(\gamma_1>0,\gamma_2\leq 0)} \\ &+ h\big(\eta_{k,t}\big)I_{(\gamma_1>0,\gamma_2>0)}\Big] + \epsilon_t' \,, \end{aligned} \tag{30}$$

leads to the following auxiliary regression for testing linearity and symmetry:

$$\hat{u}_t = \hat{\mathbf{z}}_\mathbf{1t}'\tilde{\boldsymbol{\beta}}_\mathbf{1} + \sum_{j=1}^p \beta_{2j}y_{t-j}y_{t-d} + \sum_{j=1}^p \beta_{3j}y_{t-j}y_{t-d}^2 + \sum_{j=1}^p \beta_{4j}y_{t-j}y_{t-d}^3 + v_t \,, \tag{31}$$

where v_t is a $N.I.D.(0, \sigma^2)$ process, $\tilde{\boldsymbol{\beta}}_\mathbf{1} = (\beta_{10}, \boldsymbol{\beta}_\mathbf{1}')'$, $\beta_{10} = \phi_0 - (c/4)\theta_0$, $\boldsymbol{\beta}_\mathbf{1} = \boldsymbol{\phi} - (c/4)\boldsymbol{\theta} + (1/4)\theta_0\mathbf{e}_d$, $\mathbf{e}_d = (0, 0, \ldots, 0, 1, 0, \ldots, 0)'$ with the d-th element equal to unit and $T_3(G) = f_1G + f_3G^3$ is the third-order Taylor expansion of $G(\Xi)$ at $\boldsymbol{\gamma} = \mathbf{0}$, $f_1 = \partial G(\Xi)/\partial\Xi\big|_{\boldsymbol{\gamma}=\mathbf{0}}$ and $f_3 = (1/6)\partial^3 G(\Xi)/\partial\Xi\big|_{\boldsymbol{\gamma}=\mathbf{0}}$, $G(\Xi)$ being defined in previous section. The null hypothesis is

$$H_0 : \beta_{2j} = \beta_{3j} = \beta_{4j} = 0 \;\; j = 1, \ldots, p, \tag{32}$$

The test statistic:

$$LM_1 = (SSR_0 - SSR)/\hat{\sigma_v}^2 \,, \tag{33}$$

with SSR_0 and SSR denoting the sum of squared estimated residuals from the estimated auxiliary regression (31) and under the null and alternative, respectively and $\sigma_v^2 = (1/T)SSR$, has an asymptotic χ^2_{3p} distribution under H_0.

If the model is an GESTAR(p), then $\hat{\mathbf{z}}_{\mathbf{1}} = -\mathbf{z_t}$ as in the generalized logistic case, while $\hat{\mathbf{z}}_{\mathbf{2},\mathbf{t}} = -2\boldsymbol{\theta}_{\mathbf{2}}'\mathbf{z_t}y^2_{t-2} - 2\theta_{20}y^2_{t-d} + 4c\boldsymbol{\theta}_{\mathbf{2}}'\mathbf{w_t}y_{t-d} - 2c^2\boldsymbol{\theta}_{\mathbf{2}}'\mathbf{z_t}y_t + 4c\theta_{20}y_{t-d} - 2c^2\theta_{20} = 2\hat{\mathbf{z}}^{ESTAR}_{\mathbf{2},\mathbf{t}}$, where $ESTAR$ denotes the vector $\hat{\mathbf{z}}_{\mathbf{2},\mathbf{t}}$ for the ESTAR model. That is, the vector $\hat{\mathbf{z}}_{\mathbf{2},\mathbf{t}}$ of the generalized ESTAR model is found to be two times the symmetric one. The corresponding auxiliary regression is

$$\hat{\epsilon}_t = \tilde{\boldsymbol{\beta}}_{\mathbf{1}}'\hat{\mathbf{z}}_{\mathbf{1}} + \boldsymbol{\beta}_{\mathbf{2}}'\mathbf{z_t}y_{t-d} + \boldsymbol{\beta}_{\mathbf{3}}'\mathbf{z_t}y^2_{t-d} + v_t' \,, \tag{34}$$

where v_t' is a $N.I.D.(0, \sigma^2)$ error term and $\tilde{\boldsymbol{\beta}}_{\mathbf{1}} = (\beta_{10}, \boldsymbol{\beta}_{\mathbf{1}}')'$, with $\beta_{10} = \phi_0 - c^2\theta_0$ and $\boldsymbol{\beta}_{\mathbf{1}} = \boldsymbol{\phi} - c^2\boldsymbol{\theta} + 2c\theta_0\boldsymbol{e}_d$; moreover $\boldsymbol{\beta}_{\mathbf{2}} = 2c\boldsymbol{\theta} - \theta_0\mathbf{e}_d$ and $\boldsymbol{\beta}_{\mathbf{3}} = -\boldsymbol{\theta}$. Thus the null hypothesis of linearity is

$$H_0' : \boldsymbol{\beta}_{\mathbf{2}} = \boldsymbol{\beta}_{\mathbf{3}} = 0, \tag{35}$$

which can be tested by the test statistic:

$$LM_2 = (SSR_0 - SSR)/\hat{\sigma}^2_{v1}, \tag{36}$$

where SSR_0 and SSR are the sum of squared residuals from (34) under the null and the alternative respectively, $\hat{\sigma}^2_{v1} = (1/T)SSR$. When the null is true, the statistic (36) is asymptotically χ^2_p distributed.

5 Simulation Study

5.1 Simulation design

A Monte Carlo simulation experiment is settled in order to investigate the empirical properties of the proposed asymmetry tests. We consider two different data generating processes (DGP):

$$y_{1,t}^{(i)} = 0.4y_{1,t-1}^{(i)} - 0.25y_{1,t-2}^{(i)} + (0.02 - 0.9y_{1,t-1}^{(i)} + 0.795y_{1,t-2})^{(s)}G^{(i)}(\Xi) + \epsilon_{1,t}^{(s)}\ , \quad (37)$$

and

$$y_{2,t}^{(i)} = 0.8y_{2,t-1}^{(i)} - 0.7y_{2,t-2}^{(i)} + (0.01 - 0.9y_{2,t-1}^{(i)} + 0.795y_{2,t-2}^{(i)})G^{(i)}(\Xi) + \epsilon_{2,t}^{(s)}\ , \quad (38)$$

where

$$G^{(i)}(\Xi) = \Big(1 + \exp\Big\{ - h(\eta_t)^{(i)} I_{(\gamma_1<0,\gamma_2<0)} + h(\eta_t)^{(i)} I_{(\gamma_1>0,\gamma_2<0)} + h(\eta_t)^{(i)} I_{(\gamma_1<0,\gamma_2>0)} + + h(\eta_t)^{(i)} I_{(\gamma_1>0,\gamma_2>0)}\Big]\Big\}\Big)^{-1}, \quad (39)$$

with $\epsilon_t^{(i)} \sim N(0,1)$, $i = \{1,\ldots,I\}$ denoting the i-esim simulation of the process $\{y_t\}_{t=1}^{T}$ with $s = y_{t-1}$, $c = \frac{1}{T}y_t^{(i)}$, $I = 1,000$.
$y_{2,t}^{(i)}$ (henceforth "DGP 1") is an additive nonlinear model with accentuated nonlinear behavior, due to the high autoregressive parameters driving $G(\Xi)$ and the low ones driving the linear part; this can be the case of a macroeconomic indicator affected by an unexpected shocks affecting the whole dynamics. On the other hand, $y_{2,t}^{(i)}$ (henceforth "DGP 2") describes a more balanced scenario.
In order to simulate the function $h(\eta_t)$ we use a set of values of vector $\boldsymbol{\gamma}$. The same different combinations of (γ_1,γ_2) of the two symmetry tests has been used to investigate the empirical size and the empirical power of the three diagnostic test

described in Section A.2. These combinations allow us to understand the behavior of the test in the different cases of null, medium and extreme asymmetry, respectively, as well as the effect of having different kinds of asymmetry, due to the different signs in the two γ-s. Moreover, we consider three different hypotheses for T and the size α, namely $T = \{50, 100, 300, 1000\}$ and $\alpha = \{1\%, 5\%, 10\%\}$. For each DGP we explore the possibility of both types of different functional form of asymmetry in $G(\cdot)$ and compute the corresponding statistics (33) - (36), jointly to the "Two-Step" test hypothesis corresponding to statistics (28). In this experiment, the first 100 simulations have been discarded in order to avoid the initialization effect.
For what concerns the three diagnostic tests, in the error autocorrelation test we assumed the errors of the generating process followed an AR(1) process $u_t = \rho u_{t-1} + \epsilon$, $\epsilon \sim NID(0, 1)$ and $\rho = \{0.2, 0.4\}$. In the test for no additive asymmetry we added to the previously described DGP a generalized logistic function $G_2(\boldsymbol{\gamma}^{(2)}, h(\boldsymbol{c}, y_{t-1}))$ with coefficients $\pi_0 = 0.01$, $\pi_2 = -1.8$, $\pi_3 = 1.6$, $\boldsymbol{\gamma}^{(2)} = (\gamma_3, \gamma_4) = \{(5, 2), (50, 20), (500, 200)\}$ and $\boldsymbol{\gamma}^{(1)}$ fixed at (120,70); this ensures that the behavior of the additive component remains isolated from the second; our experience shows that if higher parameter are set, the inversion become problematic. For the test for parameter constancy, the coefficients has been simulated according to a generalized logistic smooth change with $\lambda_1 = (0, 0.4, -0.25)'$ and $\lambda_2 = (0.2, -0.9, -0.795)'$. All these devices should make our simulation exercise comparable to the ET results.

5.2 Results

The results of the "All-in-one" and "Two-step" tests discussed in Section 4 for single DGP 1 and DGP 2 are reported in Table 2 and Table 3, respectively. Several findings can be easily noticed:

- The two tests have good and similar size properties. Only in large samples the two models behave in a slightly different way because of the statistics LM_2 , being its size in for DGP 1 (0.0735 at a nominal size 5%) slightly oversized with respect to DGP 2 (0.0391), while statistics LM_1 and the TS test are more consistent.

- The two tests react similarly to different DGPs: the statistics LM_1 is more powerful of LM_2, regardless to the DGP 2 as sample size grows, although the empirical power is similar for moderate asymmetry and T=100. The TS test makes an exception: under $DGP1$, the power of S_1 is very similar to LM_1 and LM_2, while, under DGP 2, the S_1 power is full when one of the slopes is 0 and the other is positive (see rows 4, 6 or 10 and 12 in Table 3). An important difference between the two scenarios is the change is scale of the empirical power under DGP 2; for example, when T=300, LM_1 statistic passes from 0.99 to 0.13 for $\alpha = 5\%$. This implies that the detection of a dynamic asymmetric movement of the series when the underlining process is not unambiguously nonlinear remains critical.

- Both the tests are quite sensitive to different couples of (γ_1, γ_2) with respect to signs and scale: the empirical power of both tests tends to decline while $\boldsymbol{\gamma}$ has opposite signs. In particular, for $\gamma_1 < 0$, the power decays up to one third (see the case of $\boldsymbol{\gamma} = (50, 10)$ for $\alpha = \%5$ in statistic LM_2 at DGP1). In any case, all the statistics requires high slopes (500, 100 and similar) to get power in low sample. Heuristically, this is justified with the fact that the Stukel function approximates a near-to-linear function for extreme negative slopes.

- The results for the three diagnostic tests deliver a similar picture, see Tables 4 and 5. Some *caveat* are required to interpret the empirical power properties: under DGP 1, all the tests have good power, in particular for serial correlation test; the test of no additive asymmetry and parameter constancy are characterized by a duality: when the two slopes are high, that is $\boldsymbol{\gamma} = (500, 200)$, the power is extreme, while it decays for low-medium asymmetry (0.21 vs 1.00 at $\alpha = 5\%$, T=100 in no additive asymmetry test, 0.44 vs 0.87 for LM_2 statistic at the same nominal and sample size for parameter constancy). On the other hand, under DGP 2, the change in scale of the power is evident only for the parameter constancy test; interestingly, the test for no serial correlation is more powerful.

6 Illustrations

6.1 Data

In this section the GSTAR model is applied to six time series, namely: the U.S. index of industrial production and unemployment (IP and UN, respectively); the yearly average of daily International Sunspot Number (YSSN), and the Canadian Lynx data (LYNX), the Entry and Exit of U.S. firms from market (ENT and EXT, respectively); the last two series are considered just to understand the behavior of the GSTAR model under symmetric and linear environments[7]. Further informations on the dataset can be found in Table 6.

We consider also the monthly average of Sunspot Number from January 1850 to December 2013 (1962 observations) for which three different kinds of data transformations are compared to link our model to the existing literature: the logarithmic (logMSSN), square root (sqrtMSSN) and the growth rate (DLMSSN); in this case, the Kalman-smoothed version of the series is available and necessary to avoid inversion problems due to the high noise. The series and the resulting (eventually, multiple) transition function(s), plotted versus time are reported in Figure 5, while the same transitions plotted versus the selected transition variable are shown in Figure 6.

[7]These last two series are available from US Census Boureau only at yearly frequency and has been disaggregated using the IP as indicator variable; see Rossi and Zanetti Chini (2016) for details.

6.2 Results

Tables 7 – 8 – 9 report the results of the modelling strategy discussed in Section 3, and specifically: the descriptive analysis of the series, using basic statistics and a battery of test for normality, ARCH-effect, serial uncorrelation, identical distribution and the t-statistics of Dikey-Fueller test augmented for two lags in the first panel; the result of the LST linearity test, the selected model according to the Teräsvirta rule and two symmetry tests introduced in Section 4 (second panel); parameter estimates and HAC standard errors of the selected GSTAR model with its equivalent symmetric specification, for which the possibility of multiple regimes has been taken in consideration (MR-STAR, third panel); the diagnostic tests (fourth panel). Informations on rolling forecasting exercise and tests on forecast comparison are shown in Tables 10 – 12.

Several facts arises:

- According to the AIO test, the dynamic asymmetry here introduced cannot be rejected for IP, UN, LYNX and YSSN, while ENT is linear and EXT is nonlinear but symmetric. However, the TS approach, which strictly follows the original Stukel's methodology changes this result; this seems reasonable at least for case of UN. This is not the case of monthly sunspots series, for which the TS test, although still not able to reject the null for the selected model, starts to reject if different data and models are used[8].

- The GSTAR transition function differs from its symmetric equivalent and this holds also when the two estimated slopes are very similar. In particular, Figure 5, panels (a), (c) and (d), shows that the estimated asymmetric $G(\boldsymbol{\gamma}, s_t, c)$ functions tends to concentrate in the upper part of the of the space of continuum of states (between 0.5 and 1); on the other side, in panel (b) of the same figure, the estimated GLSTAR transition for UN reduces of 30% from

[8] The results, not shown here, can be provided upon request.

the full [0, 1] range (from 1950 to 1980) to a [0.2, 0.9] range (after first 1980s). This means that GSTAR allows the modeller to reproduce the cyclical movements of the data and their phases better than the traditional parametrization. The differences between transition functions are still more evident in Figure 6. Panel (e) and (f) shows also the transition functions for non asymmetric series; in particular, ENT conveys still a weak nonlinear transition function. Anyway, the high standard errors of its estimated parameters make them non significant.

- The new parametrization allows the investigator to gain in terms of parsimony with respect to other symmetric counterparts. This is immediately evident in the LYNX example, where an autoregressive order 7 is sufficient to pass all the diagnostic tests, whereas more lags was required by previous literature using SETAR and STAR models (Tong, 1977; Teräsvirta, 1994). The monthly sunspot data enforces this finding. Moreover, GSTAR model is sensitive to changes in scale, so that a further transformation tends to over-smooth (exacerbate) the nonlinear dynamics of the process if further transformations are applied. In this sense, Figure 7 on monthly Sunspot series is almost self-explanatory: under square root-transformation, (panels (c) and (d)), the GSTAR transition reproduces most of the asymmetric cycle fluctuations of the data in a very precise way and its symmetric version seems almost equivalent. On the other side, if growth rates are considered (same figure, panels (e) and (f)), the GSTAR model reproduces the asymmetric dynamics of the data, but the states oscillates in a window of [0.45-0.6], leading to a quasi-linear behavior, as confirmed by the low slope coefficients; on the other side, the MR-STAR transitions are well-behaving.

- Concerning point forecasting properties reported in Table 10), the GSTAR model beats almost always its symmetric counterpart according to mRAE, while, in terms of RMSFE, the GSTAR wins in many forecast horizons of YSSN and LYNX and at longest horizons of UN; similar evidence is provided by MFE criterion: the new model prevails in two cases (UN and YSSN) whereas at very short term, the AR is still a good model for IP and LYNX. This superiority is less evident if considering sMAE: the two nonlinear specification almost equivalent for IP, while, for other three cases the MR-STAR prevails with a factor of less than 0.1%.

- In terms of density forecasting, if we look at Table 10, the GSTAR wins for only UN according to LogS, while AR does for IP and short horizons of YSSN and MR-STAR for LYNX. The GSTAR returns to outperform if QSR is used, beating its competitors for YSSN and LYNX and at long horizons of IP and UN. Differently, the CRPS conveys a clear superiority of MR-STAR, which win in almost all cases, with the exception of YSSN and short-run horizons of IP (where AR better). The qS^{α} enforces this result by providing evidence in favor linear specifications with the only exception of IP, and STAR being still the second best for IP and UN.

- Introducing uncertainty in the forecasting process changes these findings. This can be noticed in Table 11: in point forecasting, the symmetric MR-STAR beats GSTAR in all measures of IP and UN and the majority of horizons and measures of ENT and EXT; in these last two cases, an interesting exception is RMFE, where GSTAR win. On the other side, the asymmetric specifications become the best in density forecasting in IP and UN, while for ENT and EXT the estimated scores for linear and symmetric model are almost equivalent, as expected by the result of the asymmetry test.

- There is statistical evidence of an improvement in forecasting ability for dynamic asymmetric models with respect to traditional ones. Table 12 shows the result of three equal predictive ability tests. As expected by the fact that our models are nested, the Diebold-Mariano test gives counterintuitive results: the null hypothesis of no improvement in forecasting ability for a nonlinear and dynamically asymmetric specifications of ENT (EXT) strongly is (not) rejected with respect to linear (and symmetric) ones, whereas the linearity and asymmetry test suggest the converse). On the other side, the most general Giacomini-White test does not convey any evidence of improvement in forecasting ability of GSTAR model when samples are symmetric and linear, while it does when the series is proved to be strongly asymmetric (as in the case of IP and UN). This (more intuitive) finding is partially confirmed by Amisano-Giacomini test, albeit, if considering LogS, there are some exceptions: UN, where there is no evidence against STAR specification; and EXT where, despite the fact that the series is symmetric, GSTAR-based forecasts are still preferable to symmetric STAR ones .

7 Discussion and Conclusions

The Generalized Logistic function is applied to STAR family of models as simple, statistically feasible way to capture the dynamic asymmetry in the conditional mean of a time series. The resulting GSTAR model ensures the smoothness of the transition function by construction without requiring further efforts for what concerns identification and estimation.

Two test for the null of dynamic symmetry and three diagnostic tests are proposed and investigated. Our simulation results make us confident in the use of the AIO test, as well as in the use of all diagnostics, while the TS test seems us excessively conservative. In any case, such a feature remains not easy to detect if DGP is not properly imposed.

The GSTAR specification, due to its logarithmic (exponential) rescaling imposed by h-functions (6) - (7), is able to characterize some of the most prominent examples of nonlinear time series in applied sciences.

The dynamic asymmetry is an important feature to take in account for point forecasting aims. The density forecasting exercise confirm - and possibly, because of the variety of datasets used, enforce - the Kascha and Ravazzolo (2010) evidence that the relation between highest LogS and lower RMSFE is not one-to-one. In addiction to this, we find that such a relation breaks under CRPS and reverts under qS^{α}. This means that dynamic asymmetric models are not superior to traditional STARs if robust measures are used, albeit nonlinear specifications remains preferable to linear ones.

Moreover, uncertainty seems to clarify and confirm definitively this non-one-to-one relationship between RMSFE and LogS measures. Bootstrap-based forecasts by the GSTAR model are superior to the ones originated by MR-STAR and symmetric specifications are useful only if compared with linear model based forecasts. However, our forecast comparisons are based on statistical tests mainly developed for linear models and few is known about the inference on uncertain environment if dynamic asymmetry is assumed. A deeper investigation and, possibly, a development of the test by Giacomini and Rossi (2010) could be an important step in this sense. Finally, the dynamic asymmetry is postulated in the economic theory of several topics; for example, see Canepa and Zanetti Chini (2016) for an application in House Pricing. In a similar fashion, the high flexibility of the GSTAR model here demonstrated makes it a immediately available tool for further development in the literature of other fields, like Exchange Rate dynamics, Behavioral Macro-Finance and Financial Econometrics.

Acknowledgements

This paper has been mainly developed when the author was visiting PhD student at CREATES - Center for Research in Econometric Analysis of Time Series (DNRF78), funded by the Danish National Research Foundation. The hospitality and the stimulating research environment provided by Niels Haldrup are gratefully acknowledged. The author is particularly grateful to Timo Teräsvirta and Tommaso Proietti for their supervision. He also thanks Barbara Annicchiarico, Cristina Amado, Gianna Boero, Alessandra Canepa, Carolina Castagnetti, Peter Exterkate, Jan G. De Gooijer, Menelaos Karanasos, Alessandra Luati, James Morley, Eduardo Rossi, Phil Rothman and Howell Tong for helpful discussions. He is also grateful to seminar participants at "ECTS2011" Conference held in Villa Modragone, CFE 2012 in Oviedo, ICEEE-5th in Genoa, 8^{th} BMRC-QASS Conference on Macro and Financial Economics in Brunel University and the 2013 Annual Conference of Royal Statistical Society in Newcastle. This paper has been awarded of the "James B. Ramsey" Prize for the best paper in Econometrics presented by a PhD student at the 21st Annual Symposium of the Society for Nonlinear Dynamics and Econometrics held in University of Milan-Bicocca. Moreover, the Author thanks Marcelo C. Medeiros and Barbara Rossi for having published their MatLab routines in their web-pages; these codes has been fundamental to implement the results of this paper; obviously, the usual disclaimer applies. This work is in memory of Giancarlo Marini for his great support and encouragement. Finally, the Author is in debt with the doctors, assistants and nurses of the Department of Emathology of Policlinico "S. Matteo" of Pavia, without whose (free) cares this paper would probably remain unfinished.

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A Appendix

A.1 Mathematical derivations

A.1.1 Preliminar notation

Let denote $G_t = G(\Xi)$, $\Xi = [\gamma_1, \gamma_2, c]$ or $[\gamma_1, \gamma_2, c_1, c_2]$ in case of GLSTR1 and GESTAR (GESTR). Then we can re-define $G(\Xi)$ as:

$$
\begin{aligned}
G^{\{i\}}(\Xi) &= [1 + g(f^{\{i\}}(\Xi))]^j, \\
f^{\{GLSTR\}}(\Xi) &= -[h(\eta_t^L)I_{(\gamma_1 \le 0, \gamma_2 \le 0)} + h(\eta_t^L)I_{(\gamma_1 \le 0, \gamma_2 > 0)} + h(\eta_t^L)I_{(\gamma_1 > 0, \gamma_2 \le 0)} + h(\eta_t^L)I_{(\gamma_1 > 0, \gamma_2 > 0)}], \\
f^{\{GLSTR2\}}(\Xi) &= -[h(\eta_t^{2L})I_{(\gamma_1 \le 0, \gamma_2 \le 0)} + h(\eta_t^{2L})I_{(\gamma_1 \le 0, \gamma_2 > 0)} + h(\eta_t^{2L})I_{(\gamma_1 > 0, \gamma_2 \le 0)} + h(\eta_t^{2L})I_{(\gamma_1 > 0, \gamma_2 > 0)}], \\
f^{\{GESTR\}}(\Xi) &= -[h(\eta_t^E)I_{(\gamma_1 \le 0, \gamma_2 \le 0)} + h(\eta_t^E)I_{(\gamma_1 \le 0, \gamma_2 > 0)} + h(\eta_t^E)I_{(\gamma_1 > 0, \gamma_2 \le 0)} + h(\eta_t^E)I_{(\gamma_1 > 0, \gamma_2 > 0)}],
\end{aligned}
$$

with $i = \{L, 2L, E\}$, denoting the Logistic, Double Logistic and Exponential parametrization, $j = \{1; -1\}$, with $j = 1$ only if $f(\Xi) = f^{GESTR}(\Xi)$, $\eta_t^L = s_t - c$, $\eta_t^{2L} = (s_t - c_1)(s_t - c_2)$, $\eta_t^L = (s_t - c)^2$. Moreover, let $f'(\Xi) = -[h'(\eta_t)I_{(\gamma_1 \le 0, \gamma_2 \le 0)} + h'(\eta_t)I_{(\gamma_1 \le 0, \gamma_2 > 0)} + h'(\eta_t)I_{(\gamma_1 > 0, \gamma_2 \le 0)} + h'(\eta_t)I_{(\gamma_1 > 0, \gamma_2 > 0)}]$ define the first derivative of $f(\Xi)$ and $D = 1 + g(\Xi)$ denote the denominator of the fraction which is the result of the computation of the second derivatives so that:

$$
\begin{aligned}
D^2 &= 1 + g(\Xi)^2 + 2g(\Xi), \\
g^{\{i\}}(\Xi)^2 &= 1 + \exp\{-2(h(\eta_t^{\{i\}})I_{(\gamma_1 \le 0, \gamma_2 \le 0)} + h(\eta_t^{\{i\}})I_{(\gamma_1 \le 0, \gamma_2 > 0)} + h(\eta_t^{\{i\}})I_{(\gamma_1 > 0, \gamma_2 \le 0)} + \\
&\quad + h(\eta_t^{\{i\}})I_{(\gamma_1 > 0, \gamma_2 > 0)})\} + 2\exp\{h(\eta_t^{\{i\}})I_{(\gamma_1 \le 0, \gamma_2 \le 0)} + h(\eta_t^{\{i\}})I_{(\gamma_1 \le 0, \gamma_2 > 0)} + \\
&\quad + h(\eta_t^{\{i\}})I_{(\gamma_1 > 0, \gamma_2 \le 0)} + h(\eta_t^{\{i\}})I_{(\gamma_1 > 0, \gamma_2 > 0)}\},
\end{aligned}
$$

A.1.2 LSTR1 case

When the transition equation is a Generalized Logistic, we have the following derivatives:

(i)

$$G_{\gamma_1}(\Xi) = -\frac{g'(f(\Xi)) \cdot f'(\Xi)}{D^2} \tag{40}$$

where:

$$h'(\eta_t)I_{(\eta_t>0)} = \frac{\partial}{\partial\gamma_1}h(\boldsymbol{\gamma}) = \begin{cases} -\frac{1}{\gamma_1^2} \cdot \exp(|\eta_t|-1)(|\eta_t|-1) & \text{if } \ \gamma_1 > 0 \\ 0 & \text{if } \ \gamma_1 = 0 \\ -\frac{1}{\gamma_1^2} \cdot \ln(1-\gamma_1|\eta_t|) + \frac{|\eta_t|}{1-\gamma_1|\eta_t|}) & \text{if } \ \gamma_1 < 0 \end{cases} \tag{41}$$

and

$$h'(\eta_t)I_{(\eta_t\leq 0)} = \begin{cases} 0 \ \text{ if } \ \gamma_2 > 0 \\ 0 \ \text{ if } \ \gamma_2 = 0 \\ 0 \ \text{ if } \ \gamma_2 < 0 \end{cases} \tag{42}$$

(ii) $G_{\gamma_2}(\cdot)$: equal to (40) but with

$$h'(\eta_t)I_{(\eta_t>0)} = \frac{\partial}{\partial\gamma_2}h(\boldsymbol{\gamma}) = \begin{cases} 0 \ \text{ if } \ \gamma_1 > 0 \\ 0 \ \text{ if } \ \gamma_1 = 0 \\ 0 \ \text{ if } \ \gamma_1 < 0 \end{cases} \tag{43}$$

and

$$h'(\eta_t)I_{(\eta_t\leq 0)} = \begin{cases} \frac{1}{\gamma_2}\exp(1-\gamma_2|\eta_t|) \cdot (\frac{1}{\gamma_2} + |\eta_t|) & \text{if } \ \gamma_2 > 0 \\ 0 & \text{if } \ \gamma_2 = 0 \\ -\frac{1}{\gamma_2}\left[\frac{1}{\gamma_2}\ln(\gamma_2|\eta_t|-1) + \frac{|\eta_t|}{\gamma_2|\eta_t|-1}\right] & \text{if } \ \gamma_2 < 0 \end{cases} \tag{44}$$

(iii) $G_c(\cdot)$: equal to (40) but with

$$f'(\Xi) = h(\eta_t)I_{(\eta_t\leq 0)} + h(\eta_t)I_{(\eta_t>0)} \tag{45}$$

A.1.3 LSTR2 case

When the transition equation is a (Generalized) Double Logistic as in model (**??**), we have the following derivatives:

(i) $G_{\gamma_1}(\cdot)$ equal to equation (40) with: $f^{\{i\}}(\Xi) = f^{\{GLSTR2\}}(\Xi) = -[h(\eta_t^{2L})I_{(\gamma_1\leq 0,\gamma_2\leq 0)} + h(\eta_t^{2L})I_{(\gamma_1\leq 0,\gamma_2>0)} + h(\eta_t^{2L})I_{(\gamma_1>0,\gamma_2\leq 0)} + h(\eta_t^{2L})I_{(\gamma_1>0,\gamma_2>0)}]$, $g^{\{GLSTR2\}} = \exp\{f^{\{GLSTR2\}}(\Xi)\}$, $h'(\eta_t^{2L})I_{(\eta_t^{2L}>0)}$ and $h'(\eta_t^{2L})I_{(\eta_t^{2L}\leq 0)}$ equal to systems (41) and (42).

(ii) $G_{\gamma_2}(\cdot)$: equal to equation (40) with: $f^{\{GLSTR2\}}(\Xi)$ and $g^{\{GLSTR2\}}(\Xi)$ above defined as in case (i) and $h'(\eta_t^{2L})I_{(\eta_t^{2L}\geq 0)}$ and $h'(\eta_t^{2L})I_{(\eta_t^{2L}<0)}$ equal to systems (43) and (44) respectively.

(iii) $G_{c_1}(\cdot)$: equal to equation (40) with: $f^{\{GLSTR2\}}(\Xi)$ and $g^{\{GLSTR2\}}(\Xi)$ defined as in case (i) and

$$f'(\Xi) = h'(\eta_t^{2L})I_{(\eta_t^{2L}\leq 0)}(s_t - c_2) + h'(\eta_t^{2L})I_{(\eta_t^{2L}>0)}(s_t - c_2) \tag{46}$$

(iv) $G_{c_2}(\cdot)$: equal to equation (40) with: $f^{\{GLSTR2\}}(\Xi)$ and $g^{\{GLSTR2\}}(\Xi)$ defined as in case $i)$ and

$$f'(\Xi) = h'(\eta_t^{2L})I_{(\eta_t^{2L}\leq 0)}(s_t - c_1) + h'(\eta_t^{2L})I_{(\eta_t^{2L}>0)}(s_t - c_1) \tag{47}$$

A.1.4 ESTR case

When the transition equation is an exponential as in model (**??**), we have: $f^{\{ESTR\}}(\Xi) = -[h(\eta_t^E)I_{(\gamma_1\leq 0,\gamma_2\leq 0)} + h(\eta_t^E)I_{(\gamma_1\leq 0,\gamma_2>0)} + h(\eta_t^E)I_{(\gamma_1>0,\gamma_2\leq 0)} + h(\eta_t^E)I_{(\gamma_1>0,\gamma_2>0)}]$, $g^{\{ESTR\}}(\Xi) = -\exp\{f^{\{E\}}\}$, hence the following derivatives:

(i) $G_{\gamma_1}(\cdot) = f^{\{ESTR\}*'(\Xi)}$ with: $f'(\Xi) = -[h(\eta_t^E)I_{(\eta_t^E\leq 0)}(s_t - c)^2 + h'(\eta_t^E)I_{(\eta_t^E\leq 0)}(s_t - c)^2)$, $h'(\eta_t^E)I_{(\eta_t^E>0]}$ and $h'(\eta_t^E)I_{(\eta_t^E\leq 0)}$ being the same of systems (41) and (42).

(ii) $G_{\gamma_2}(\cdot)$: same as $G_{\gamma_1}(\cdot)$ with $h'(\eta_t^E)I_{(\eta_t^E>0]}$ and $h'(\eta_t^E)I_{(\eta_t^E\leq 0)}$ being the same of systems (43) and (44).

(iii) $G_c(\cdot) = f^{\{ESTR\}*'}(\Xi)$ with $f'(\Xi) = h'(\eta_t^E)I_{(\eta_t^E\leq 0)}(2c) + h'(\eta_t^E)I_{(\eta_t^E>0)}(2c)$, with $h'(\eta_t^E)I_{(\eta_t^E>0]}$ and $h'(\eta_t^E)I_{(\eta_t^E\leq 0)}$ being the same of systems (43) and (44).

A.2 Diagnostics Tests

For what concerns the diagnostics, the new parametrization can be applied directly to the three tests developed by ET, which will be discussed in detail.

A.2.1 Serial independence

Consider the general additive model (1), where:

$$\epsilon_t = a'v_t + u_t = \sum_{j=1}^{q} a_j L^j \epsilon_t + u_t, \quad u_t \sim I.I.D.(0, \sigma^2), \tag{48}$$

with L^j denoting the lag operator, $v_t = (u_{t-1}, \ldots, u_{t-q})'$, $a = (a_1, \ldots, a_q)'$, $a_q \neq 0$. Under the assumption of stationarity and ergodicity (see Section 2), the null hypothesis of serial independence is $H_0 : a = 0$. By pre-multiplying eq. (2) by $1 - \sum_{j=1}^{q} a_j L^j$ we get:

$$y_t = \sum_j a_j L^j y_t + \boldsymbol{\phi}'\mathbf{z_t} - \sum_j a_j L^j \boldsymbol{\phi}'\mathbf{z_t} + \boldsymbol{\theta}'\mathbf{z_t} G(\cdot) - \sum_j a_j \boldsymbol{\theta}' G(\cdot) + \epsilon_t \ , \tag{49}$$

hence, assuming the necessary initial values $y_0, y_{-1}, \ldots, y_{-(p+q)+1}$ fixed, the pseudo normal loglikelihood for $t = 1, \ldots, T$ is:

$$\begin{aligned} \mathcal{L}_t &= constant + \frac{1}{2}\ln\sigma^2 - \frac{\epsilon_t^2}{2\sigma^2}, \\ \epsilon_t &= y_t - \sum_j a_j L^j y_t - \boldsymbol{\phi}'\mathbf{z_t} + \sum_j a_j L^j \boldsymbol{\phi}'\mathbf{z_t} - \boldsymbol{\theta}' G(\mathbf{z_{t-j}}, \Xi) + \sum_j a_j \boldsymbol{\theta}' G(\mathbf{z_{t-j}}, \Xi) \ . \end{aligned} \tag{50}$$

Consistently with the model initial assumptions, the information matrix is block diagonal, hence we can consider σ^2 fixed for the rest of the derivations. So we have:

$$\frac{\partial \mathcal{L}_t}{\partial a_j} = \frac{\epsilon_t}{\sigma^2}[y_{t-j} - \boldsymbol{\phi}'\mathbf{z_{t-j}} - \boldsymbol{\theta}' G(\mathbf{z_{t-j}}, \Xi)] \tag{51}$$

$$\frac{\partial \mathcal{L}_t}{\partial \Xi} = \frac{\epsilon_t}{\sigma^2}\left[\boldsymbol{\theta}'\mathbf{z_t}\frac{\partial G(\mathbf{z_{t-j}}, \Xi)}{\partial \Xi} - \sum_j a_j \boldsymbol{\theta}' \frac{\partial G(\mathbf{z_{t-j}}, \Xi)}{\partial \Xi}\right] . \tag{52}$$

Under H_0, consistent estimators of (51) - (52) are:

$$\left.\frac{\partial \hat{\mathcal{L}}_t}{\partial a_t}\right|_{H_0} = \frac{1}{\sigma^2}\mathbf{\hat{u}_t \hat{v}_t} \quad \left.\frac{\partial \hat{\mathcal{L}}_t}{\partial \boldsymbol{\Xi_t}}\right|_{H_0} = -\frac{1}{\sigma^2}\mathbf{\hat{u}_t \hat{z}_t}\ , \tag{53}$$

where $\mathbf{\hat{u}_t} = (\mathbf{\hat{v}_{t-1}}, \ldots, \mathbf{\hat{v}_{t-q}})'$, $\mathbf{\hat{v}_{t-j}} = y_{t-j} - \boldsymbol{\phi}'\mathbf{z_{t-j}} - \boldsymbol{\theta}' G(\mathbf{z_{t-j}}, \hat{\boldsymbol{\Xi}})$, $j = 1, \ldots, q$, $\hat{\boldsymbol{\Xi}}$ is the QMLE of $\boldsymbol{\Xi}$ and $\mathbf{\hat{z}_t} = \frac{\partial G(\mathbf{z_t}, \hat{\boldsymbol{\Xi}})}{\partial \hat{\boldsymbol{\Xi}}} = \boldsymbol{k_t^G} = [\boldsymbol{\theta}'\mathbf{z_t} G_{\gamma 1}, \boldsymbol{\theta}'\mathbf{z_t} G_{\gamma 2}, \boldsymbol{\theta}'\mathbf{z_t} G_{\gamma c}]$. The resulting LM statistic is:

$$LM = \frac{1}{\hat{\sigma}}\left(\mathbf{\hat{u}_t'\hat{v}_t}\right)\left\{\mathbf{\hat{v}_t'\hat{v}_t} - \mathbf{\hat{v}_t'\hat{z}_t}\left(\mathbf{\hat{z}_t\hat{z}_t'}\right)^{-1}\mathbf{\hat{z}_t'\hat{v}_t}\right\}^{-1}\left(\mathbf{\hat{v}'\hat{u}_t}\right), \tag{54}$$

with $\hat{\sigma}^2 = \frac{1}{T}\sum_t u_t^2$. Under the null hypothesis, statistics (54) is asymptotically χ^2_q distributed. The partial derivatives of $G(\cdot)$ are shown in Appendix A.1. Another possibility is to use the same three-step procedure for carrying an F-test:

1. Estimate the GSTR model under the assumption of uncorrelated errors and compute the residual sum of squares $SSR_0 = \sum_{t=1}^{T} \hat{u}_t^2$.

2. Regress $\hat{u}_t$ on $\hat{v}_t$, $\mathbf{z_t}$, $\mathbf{z_t}\hat{G}(\mathbf{z_{t-d}})$, $\hat{G}_{\gamma 1}$, $\hat{G}_{\gamma 2}$, $\hat{G}_c$ (eventually $\hat{G}_{c2}$ in case of GLSTR2) and compute SSR;

3. Compute the test statistic $F_{LM} = \frac{SSR_0 - SSR}{q} / \frac{SSR}{T-n-q}$, where $n = \dim(\mathbf{\hat{z}_t})$

The $F-$statistics is preferable to the χ^2 statistics which may suffer from size problems when the number of lags is high and time series is short, so that the estimated residuals can be non-orthogonal to the gradient vector $\mathbf{\hat{z}_t}$. In this case ET suggests to add an extra-step to the step (i), consisting in regressing the estimated errors to $\mathbf{z_t}$, $\mathbf{z_t}$, $\mathbf{z_t}\hat{G}(\mathbf{z_{t-j}})$, $\hat{G}_{\gamma 1}$, $\hat{G}_{\gamma 2}$, $\hat{G}_c$; the resulting errors $\tilde{u}_t$ is used to derive the $SSR_1 = \sum_{t=1}^{T} \tilde{u}_t^2$.

A.2.2 No remaining asymmetry

As in the symmetric STAR model, we are interested to detect possible misspecification. In this case there are two plausible issues to investigate: neglected (additive) nonlinearity and, in our case, neglected asymmetry. Consider the additive GSTAR model:

$$y_t = \boldsymbol{\phi}'\mathbf{z_t} + \boldsymbol{\theta}'\mathbf{z_t}G_1(\boldsymbol{\gamma}, h(\eta_t)^{(1)}) + \boldsymbol{\pi}'\mathbf{z_t}G_2(\boldsymbol{\gamma}, h(\eta_t)^{(2),EZC}) + u_t, \tag{55}$$

with $u_t \sim I.I.D.\ (0, \sigma^2)$. The null of neglected asymmetry is:

$$H_0 : h(\eta_t)^{(2),EZC} = 0 \quad \text{vs} \quad H_0 : h(\eta_t)^{(2),EZC} \neq 0. \tag{56}$$

If $\boldsymbol{\gamma}$ is found being not null, the investigator can easily check if the additive nonlinear part is significant. The EZC version of $h(\eta_t)$ is necessary in order to nest the discussion to ET framework. We assume that, under H_0, $\boldsymbol{\Xi}$ can be consistently estimated by QML. Similarly to the symmetric case, it should be noticed that the model is not identified under H_0, so that the Taylor expansion of the $G(\cdot)$ suggested by LST can be used in order to circumvent this problem. In this case, we assume $G_2(\cdot)$ as generalized logistic and replace it with its third-order Taylor expansion about $h(\boldsymbol{\gamma})^{(2)} = 0$. This implies:

$$T_2 = g_{20} + g_{21}y_{t-l} + g_{22}y_{t-l}^2 + g_{23}y_{t-l}^3\ , \tag{57}$$

where g_{2j}, $j = 0, 1, 2, 3, 4$ are functions of $\boldsymbol{\gamma}^{(2)}$ such that $g_{20} = g_{21} = g_{22} = g_{23} = 0$ for $\boldsymbol{\gamma}^{(2)} = \mathbf{0}$, consistently with the definition of $h_\gamma(s_t)$. By re-parametrizing, the model (55) became:

$$y_t = \beta_0'\mathbf{z_t} + \boldsymbol{\theta}'\mathbf{z_t}G_1(\cdot) + \beta_1'\tilde{\mathbf{z}}_\mathbf{t}y_{t-l} + \beta_2'\tilde{\mathbf{z}}_\mathbf{t}y_{t-l}^2 + \beta_3'\tilde{\mathbf{z}}_\mathbf{t}y_{t-l}^3 + r_t\ , \tag{58}$$

where $\tilde{\mathbf{z}}_\mathbf{t} = (y_{t-1}, \ldots, y_{t-p})'$. The null hypothesis of no additive nonlinearity is $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$, and, as in the symmetric case, under H_0, $r_t = u_t$. The LM statistics distributes as a $\chi^2(3p)$. As in the symmetric case, the test preserves power also against generalized exponential transition. Since there are no modifications in the statistical assumptions concerning the errors distribution, the asymptotic theory is the same of the symmetric STAR case. The test statistic is (29) with $\hat{\mathbf{z}}_\mathbf{t} = (\mathbf{z_t}, \mathbf{z_t}\hat{G}(\cdot), \hat{G}_{\gamma 1}, \hat{G}_{\gamma 2}, \hat{G}_c)'$ (or $\hat{G}_{c1}$, $\hat{G}_{c2}$ in case of GLSTR2), whereas $\mathbf{v_t} = (\tilde{\mathbf{z}}_\mathbf{t}' y_{t-l}, \tilde{\mathbf{z}}_\mathbf{t}' y_{t-l}^2, \tilde{\mathbf{z}}_\mathbf{t}' y_{t-l}^3)'$. As in the symmetric STAR model, the test is implemented with the same procedure for serial correlation, the F-test has $(3p)$ and $(T-n-3p)$ degrees of freedom and the Teräsvirta rule can be applied to (58) in order to select the form of the transition. If this selection is not desirable, a polynomial expansion of (55) can be performed to build up an omnibus test, but in this case, a rejection of the null of no additive nonlinearity will not give any qualitative information, that is why we do not take in consideration this scenario.

A.2.3 Parameter constancy

Consider the model:

$$y_t = \boldsymbol{\phi}(\boldsymbol{t})'\bar{\mathbf{z}}_\mathbf{t} + \boldsymbol{\theta}(\boldsymbol{t})'\tilde{\mathbf{z}}_\mathbf{t} G(\boldsymbol{\gamma}, h(\eta_t)) + u_t, \quad u_t \sim I.I.D.\ (0, \sigma^2)\ , \tag{59}$$

with $\bar{\mathbf{z}}_\mathbf{t}$ denoting the $k \leq p+1$ element of $\mathbf{z_t}$ for which the corresponding element of ϕ is not assumed zero a priori, $\tilde{\mathbf{z}}_\mathbf{t}$ is the same $(l \times 1)'$ for the element of $\boldsymbol{\theta}$. Let $\tilde{\boldsymbol{\phi}}$ and $\tilde{\boldsymbol{\theta}}$ denote the equivalent $(k+1)$ and $(l+1)$ parameter vectors, $\boldsymbol{\phi}(\boldsymbol{t}) = \tilde{\boldsymbol{\phi}} + \lambda_1 G_j(t; \boldsymbol{\gamma}, h(\eta_t)^{(1)})$, and $\boldsymbol{\theta}(\boldsymbol{t}) = \tilde{\boldsymbol{\theta}} + \lambda_2 G_j(t; \boldsymbol{\gamma}, h(\eta_t)^{(2)})$ with λ_1 and λ_2 being a $(k \times 1)$ and $(l \times 1)$ vectors respectively. Then the null of parameter constancy in (59) is

$$H_0 : G_j(t; \boldsymbol{\gamma}, h(\eta_t)) \equiv 0 \ \ (\text{or} \equiv \text{const}). \tag{60}$$

Three forms for G_j can be considered:

$$
\begin{aligned}
G_1(t;\boldsymbol{\gamma},h(\mathbf{c},s_t)) &= (1+\exp\{-h(\eta_t^{GL})\})^{-1} \text{ with} \\
\eta_t^{GL} &\equiv t-c\,, \\
G_2(t;\boldsymbol{\gamma},h(\mathbf{c},s_t)) &= 1+\exp\{-h(\eta_t^{GE})\} \text{ with} \\
\eta_t^{GE} &\equiv (t-c)^2\,, \\
G_3(t;\boldsymbol{\gamma},h(\mathbf{c},s_t)) &= (1+\exp\{-h(\eta_t^{C})\})^{-1} \text{ with} \\
\eta_t^{C} &\equiv (t^3+c_{12}t^2+c_{11}t+c_{10})
\end{aligned}
\tag{61}
$$

The null of parameter constancy is $H_0:\boldsymbol{\gamma}=\mathbf{0}$. Notice that in this case the model is identified also in case of $\boldsymbol{\gamma}<\mathbf{0}$, so that the only identifying restriction is that $\boldsymbol{\gamma}\neq\mathbf{0}$. G_1 and G_2 are the Generalized Logistic and Exponential smooth transition of the change in parameters, while G_3 is a cubic function which allows for both monotonically and non-monotonically changing parameters and can be seen as a general case of G_1 and G_2 when building up a test. As suggested by the literature, we use a third-order Taylor expansion of G_3 about $\boldsymbol{\gamma}=\mathbf{0}$:

$$
T_3(t;\boldsymbol{\gamma},h(\eta_t)) = \frac{1}{4}h(\boldsymbol{\gamma})(t^3+c_{12}t^2+c_{11}t+c_{10})+R(t,\boldsymbol{\gamma},h(\eta_t)). \tag{62}
$$

in order to approximate $\boldsymbol{\phi(t)}$ and $\boldsymbol{\theta(t)}$ in (59) using (62). This yields to:

$$
\begin{aligned}
y_t &= \boldsymbol{\beta_0'}(\bar{\mathbf{z}}_\mathbf{t})+\boldsymbol{\beta_1'}(t\bar{\mathbf{z}}_\mathbf{t})+\boldsymbol{\beta_2'}(t^2\bar{\mathbf{z}}_\mathbf{t})+\boldsymbol{\beta_3'}(t^3\bar{\mathbf{z}}_\mathbf{t})+ \\
&+\{\boldsymbol{\beta_4'}(\tilde{\mathbf{z}}_\mathbf{t})+\boldsymbol{\beta_5'}(t\tilde{\mathbf{z}}_\mathbf{t})+\boldsymbol{\beta_6'}(t^2\tilde{\mathbf{z}}_\mathbf{t})+\boldsymbol{\beta_7'}(t^3\tilde{\mathbf{z}}_\mathbf{t})\}G(t;\boldsymbol{\gamma},h(\eta_t))+r_t^*\,,
\end{aligned}
\tag{63}
$$

where $r_t^*=u_t+R(t;\boldsymbol{\gamma},h(\eta_t))$. Under H_0, $r_t^*=u_t$. In (63), $\boldsymbol{\beta_j}=h(\eta_t)\bar{\boldsymbol{\beta}}$, $j=1,\ldots,7$, hence the null hypothesis in terms of (63) becomes $H_0:\boldsymbol{\beta_j}=\mathbf{0}$, $j=1,\ldots,7$. Consequently, the locally approximated pseudo normal log-likelihood under

H_0 (ignoring R) is

$$\mathcal{L}_t = const - \frac{1}{2}\ln\sigma^2 - \frac{1}{2}\sigma^2[y_t - \boldsymbol{\beta_0'}\mathbf{w_t} - \boldsymbol{\beta_1'}(t\bar{\mathbf{w}}_\mathbf{t}) - \boldsymbol{\beta_2'}(t^2\bar{\mathbf{w}}_\mathbf{t}) - \boldsymbol{\beta_3'}(t^3\bar{\mathbf{w}}_\mathbf{t}) - \\ -\{\boldsymbol{\beta_4'}(\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_5'}(t\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_6'}(t^2\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_7'}(t^3\tilde{\mathbf{w}}_\mathbf{t})\}G(y_{t-d};\boldsymbol{\gamma}, h(\eta_t)]^2. \quad (64)$$

The partial derivatives are:

$$\frac{\partial\mathcal{L}_t}{\partial\boldsymbol{\beta_j}} = \frac{1}{\sigma^2}u_t(t^j\bar{\mathbf{w}}_\mathbf{t}), \qquad \text{j=0, \dots, 3}, \quad (65)$$

$$\frac{\partial\mathcal{L}_t}{\partial\boldsymbol{\beta_j}} = \frac{1}{\sigma^2}u_t(t^j\tilde{\mathbf{w}}_\mathbf{t})G(y_{t-d};\boldsymbol{\gamma}, h(\eta_t)), \qquad \text{j=4, \dots, 7}, \quad (66)$$

$$\frac{\partial\mathcal{L}_t}{\partial\gamma_1} = \frac{1}{\sigma^2}u_t\{\boldsymbol{\beta_4'}(\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_5'}(t\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_6'}(t^2\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_7'}(t^3\tilde{\mathbf{w}}_\mathbf{t})\}G_{\gamma_1}, \quad (67)$$

$$\frac{\partial\mathcal{L}_t}{\partial\gamma_2} = \frac{1}{\sigma^2}u_t\{\boldsymbol{\beta_4'}(\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_5'}(t\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_6'}(t^2\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_7'}(t^3\tilde{\mathbf{w}}_\mathbf{t})\}G_{\gamma_2}, \quad (68)$$

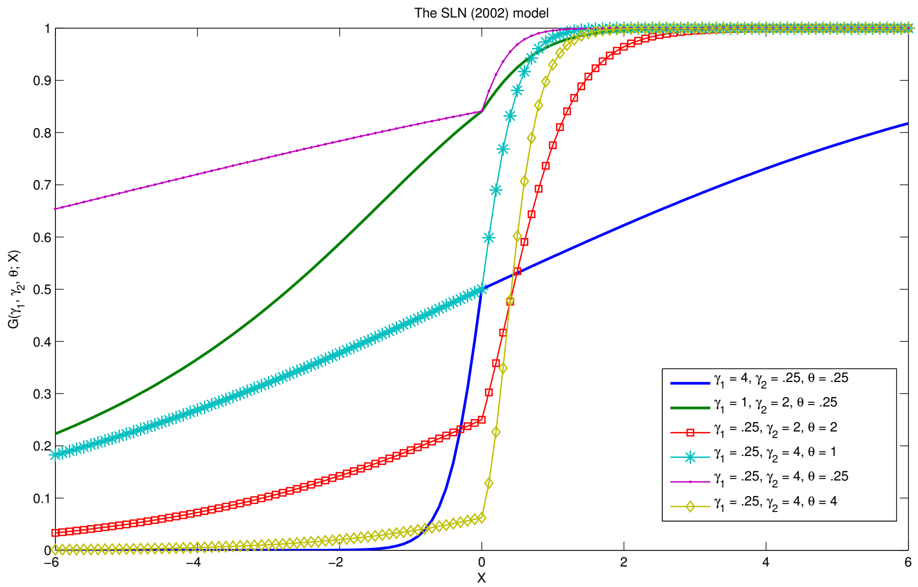
$$\frac{\partial\mathcal{L}_t}{\partial c} = \frac{1}{\sigma^2}u_t\{\boldsymbol{\beta_4'}(\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_5'}(t\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_6'}(t^2\tilde{\mathbf{w}}_\mathbf{t}) + \boldsymbol{\beta_7'}(t^3\tilde{\mathbf{w}}_\mathbf{t})\}G_c, \quad (69)$$

where G_{γ_1}, G_{γ_2}, G_c are the derivatives of $G(y_{t-d}, \boldsymbol{\gamma}, h(\eta_t))$ with respect to γ_1, γ_2 and c. With this notation, the estimators of $\frac{\partial\mathcal{L}_t}{\partial\gamma_1}$, $\frac{\partial\mathcal{L}_t}{\partial\gamma_2}$ and $\frac{\partial\mathcal{L}_t}{\partial c}$ are $\frac{\partial\hat{\mathcal{L}}_t}{\partial\gamma_1} = \frac{1}{\hat{\sigma}^2}u_t\hat{G}_{\gamma_1}$, $\frac{\partial\hat{\mathcal{L}}_t}{\partial\gamma_2} = \frac{1}{\hat{\sigma}^2}u_t\hat{G}_{\gamma_2}$, $\frac{\partial\hat{\mathcal{L}}_t}{\partial c} = \frac{1}{\hat{\sigma}^2}u_t\hat{G}_c$ respectively, so that: $\hat{\mathbf{z}}_\mathbf{t} = (1, \bar{\mathbf{z}}_\mathbf{t}', \tilde{\mathbf{z}}_\mathbf{t}'\hat{G}(y_{t-d};\cdot), \hat{G}_{\gamma_1}, \hat{G}_{\gamma_2}, \hat{G}_{\gamma_c})'$ and $\hat{u}_t = (t\bar{\mathbf{z}}_\mathbf{t}', t^2\bar{\mathbf{z}}_\mathbf{t}', t^3\bar{\mathbf{z}}_\mathbf{t}', t\tilde{\mathbf{z}}_\mathbf{t}'\hat{G}(y_{t-d},\cdot), t^2\tilde{\mathbf{z}}_\mathbf{t}'\hat{G}(y_{t-d},\cdot), t^3\tilde{\mathbf{z}}_\mathbf{t}'\hat{G}(y_{t-d},\cdot))$. Like in the symmetric scenario, under H_0, the statistic (54) has a χ^2 distribution with $3(k+l)$ degrees of freedom and the equivalent F-distribution has $3(k+l)$ and $T-4(k+l)-2$ degrees of freedom (the stastistic is denoted LM_3). The following rule is used: if H_1 is (59) with transition function G_3, then (54) is based on (63) assuming $\boldsymbol{\beta_3} = \mathbf{0}$ and $\boldsymbol{\beta_7} = \mathbf{0}$ (statistic LM_2) and, if the same alternative hypothesis has the transition function G_2, the test is based on (63), assuming $\boldsymbol{\beta_2} = \boldsymbol{\beta_3} = \mathbf{0}$ and $\boldsymbol{\beta_6} = \boldsymbol{\beta_7} = \mathbf{0}$ (statistic LM_2).

A.3 Tables and Graphs

Figure 1: Transition function for different parametrizations of Asymmetric STAR.

(a) Sollis et al. (2002)



(b) Sollis et. al (1999) - Lundbergh and Terasvirta (2006)

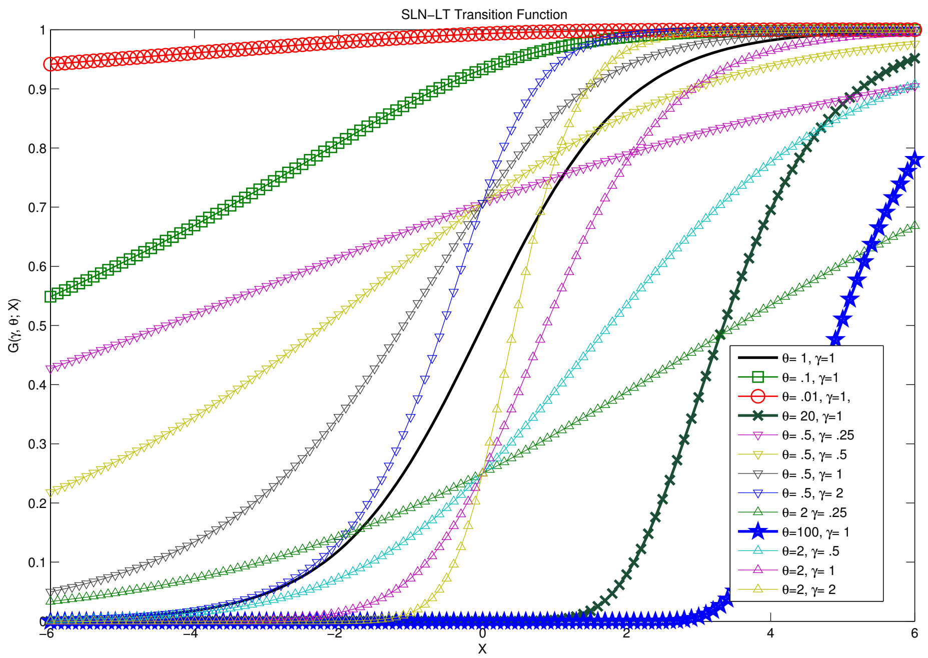


Figure 2: The Generalized Logistic function.

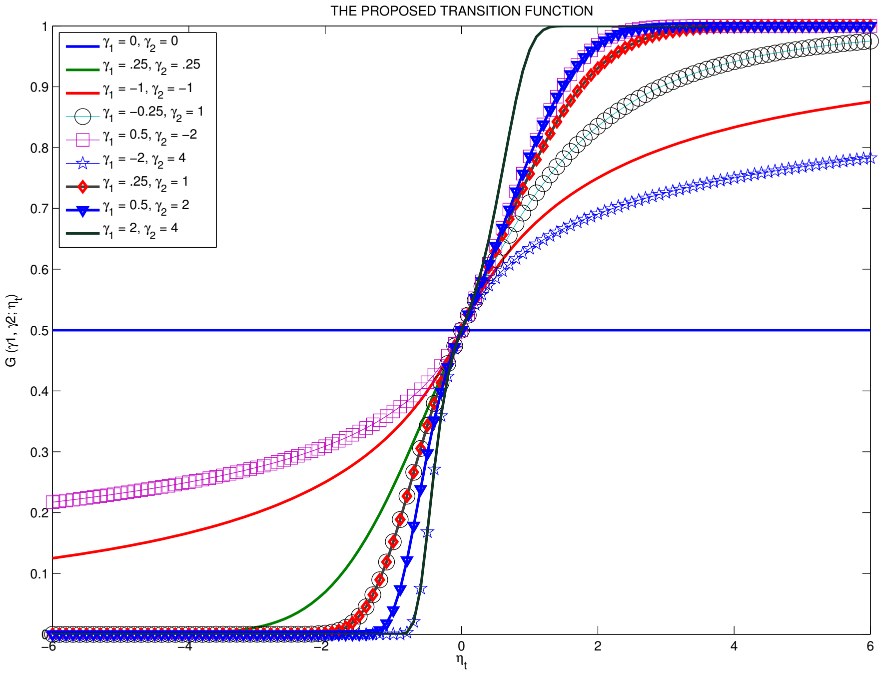


Figure 3: The Generalized Second-Order Logistic function.

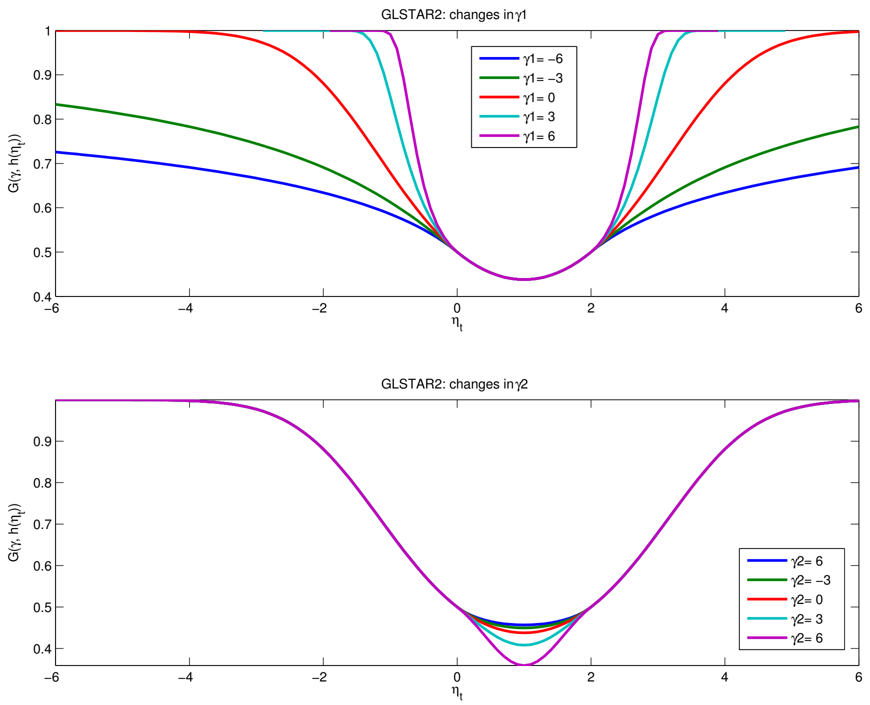
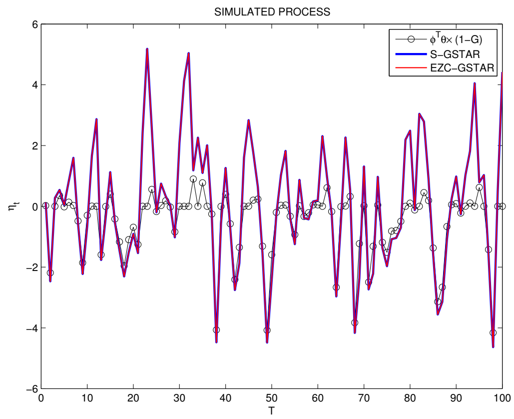
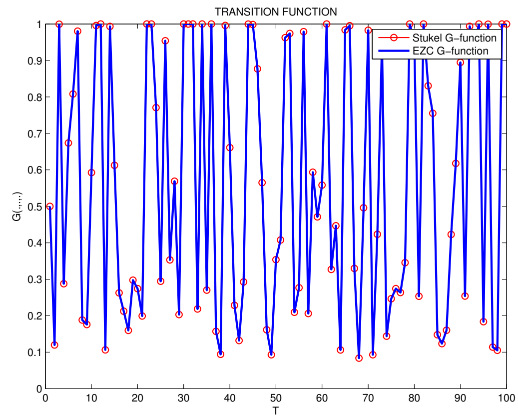


Figure 4: An example of GLSTAR model.

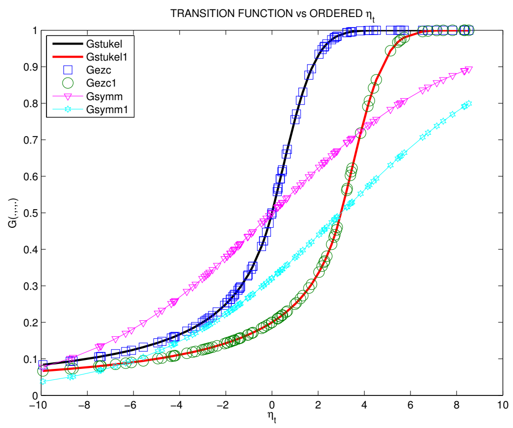
(a) Simulated process



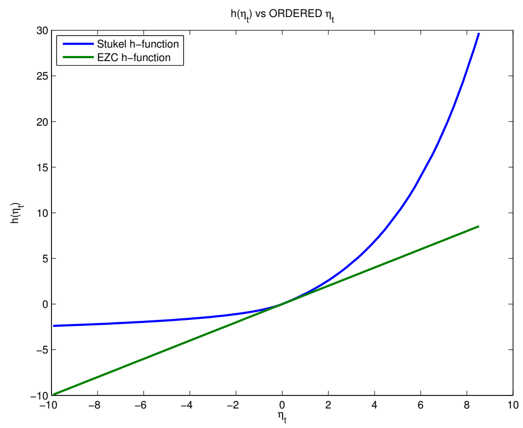
(b) Transition function



(c) Transition functions vs η_t



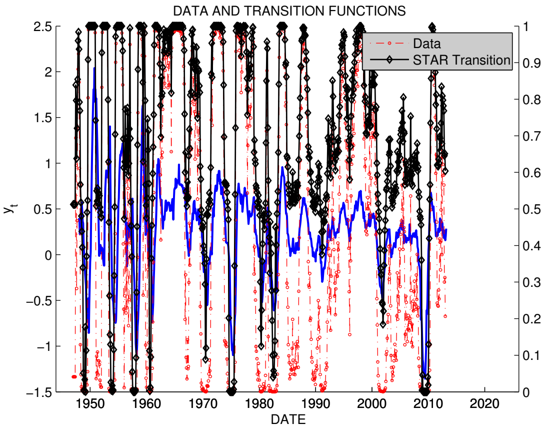
(d) The rescaling effect of h(η_t)



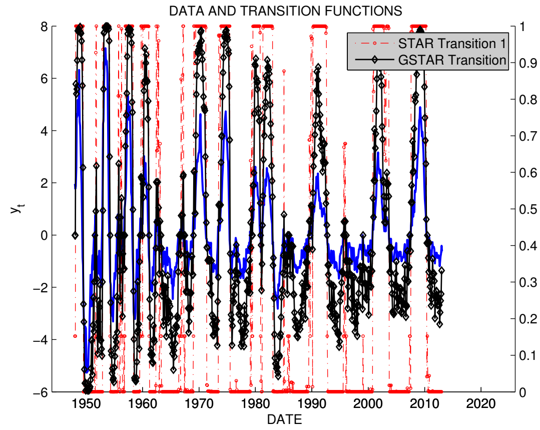
Simulation performed with following parameters: $\phi_0 = 0.05$; $\phi_1 = 0.4$; $\phi_2 = 0.25$; $\theta_0 = 0.2$; $\theta_1 = 0.4$; $\theta_2 = 0.25$; $\gamma_1 = 0.25$; $\gamma_2 = -1.0$; $c = 0$; $c_1 = 3$; $c_2 = 5$; $T = 100$

Figure 5: Estimated transition function for (MR)STAR and GSTAR specifications.

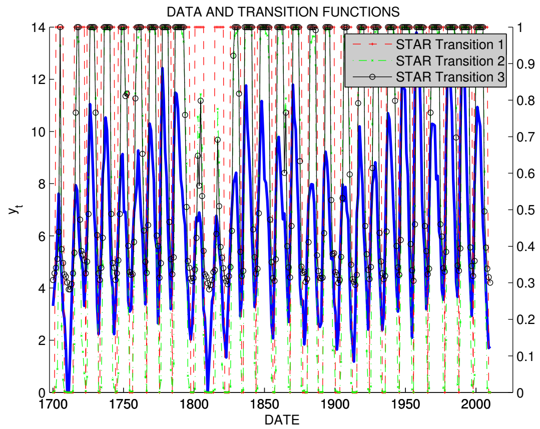
(a) U.S. Industrial Production



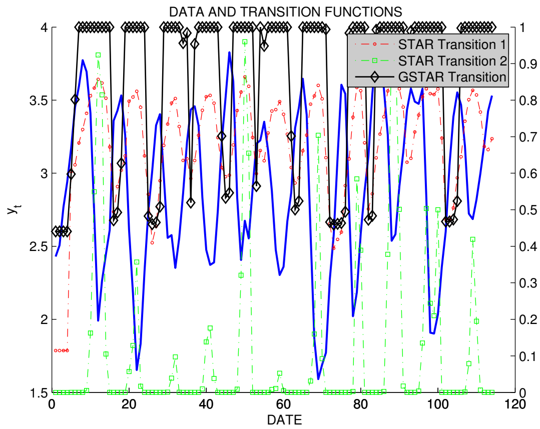
(b) U.S. Unemployment



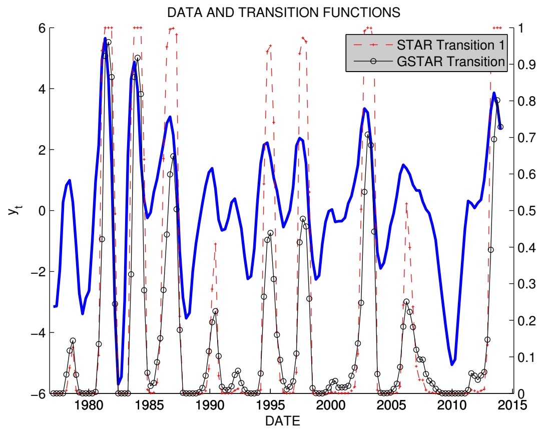
(c) Yearly Sunspot Number



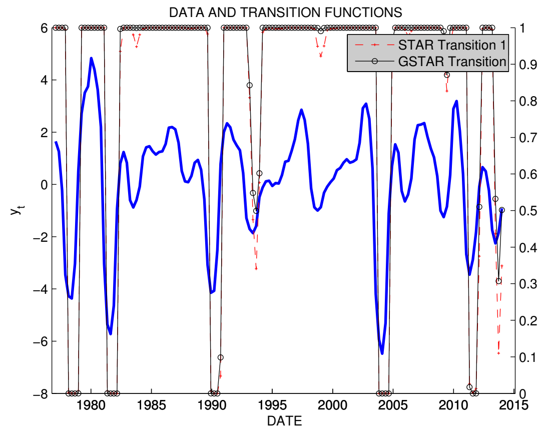
(d) Canadian Lynx



(e) U.S. Enstablishment Entry



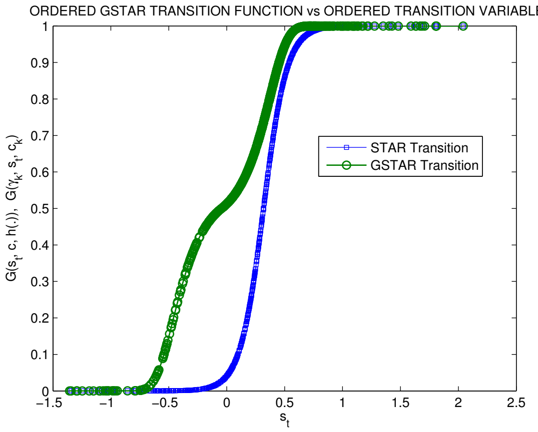
(f) U.S. Enstablishment Exit



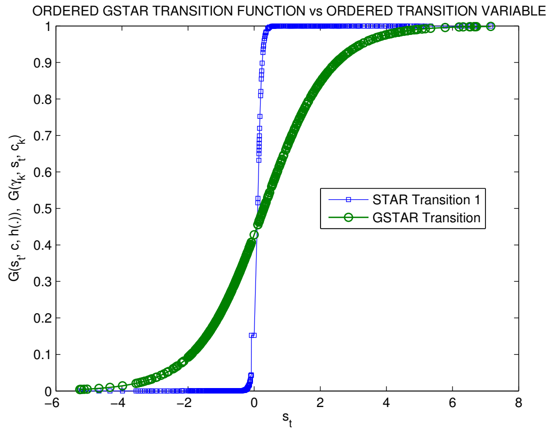
NOTE: The data are plotted in blue line.

Figure 6: Estimated transition functions vs transition variable.

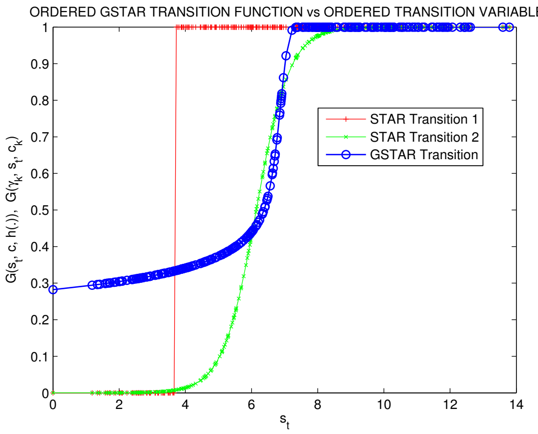
(a) U.S. Industrial Production



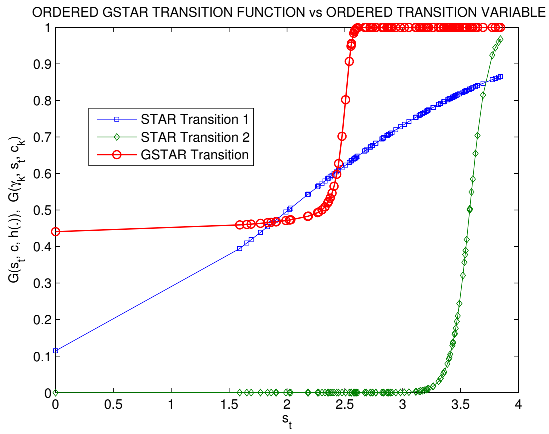
(b) U.S. Unemployment



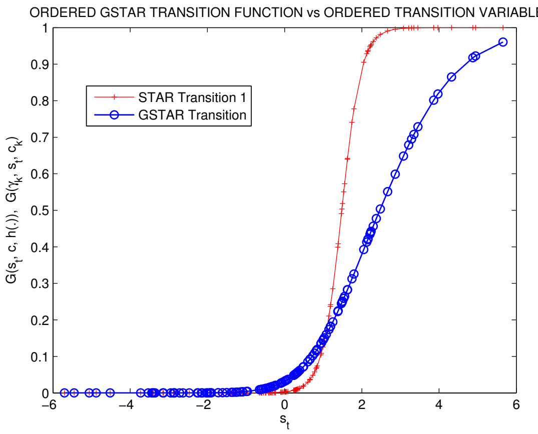
(c) Yearly Sunspot Number



(d) Canadian Lynx



(e) U.S. Enstablishment Entry



(f) U.S. Enstablishment Exit

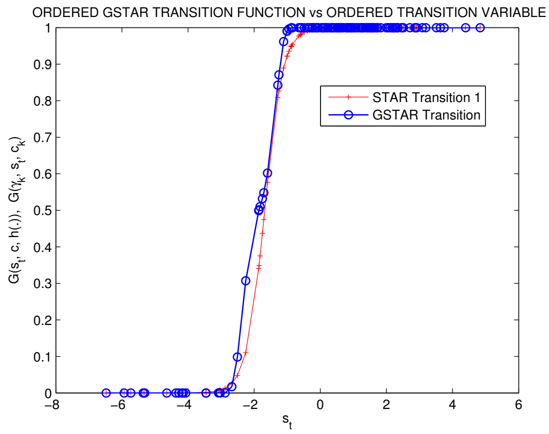
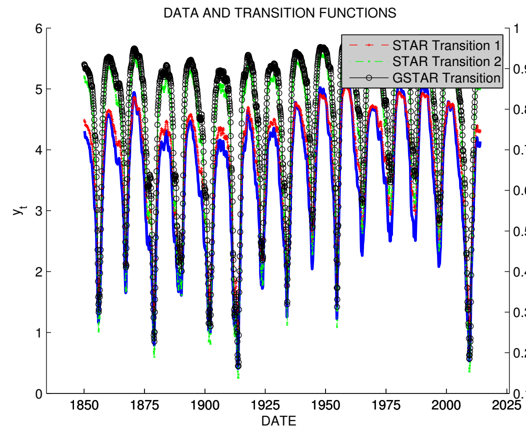
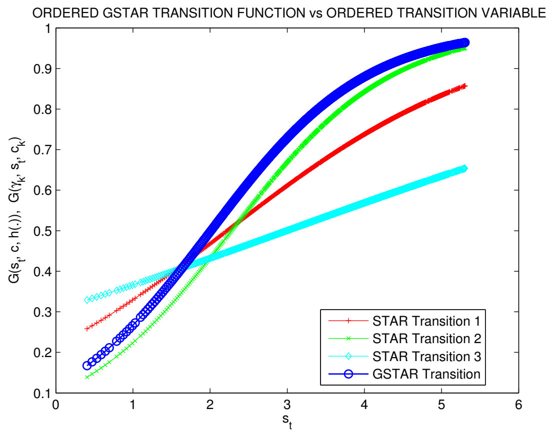


Figure 7: Monthly SSN: estimated transition functions for different data transformations.

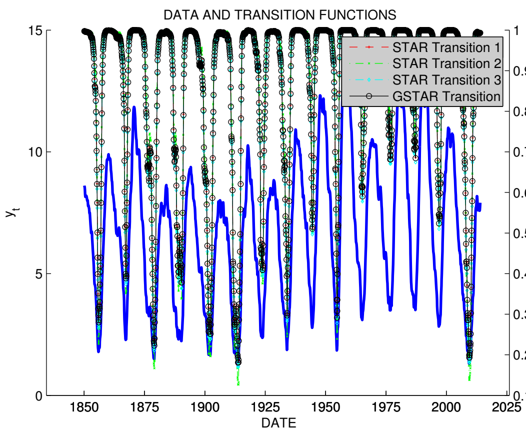
(a) logMSSN: G(Ξ) vs t



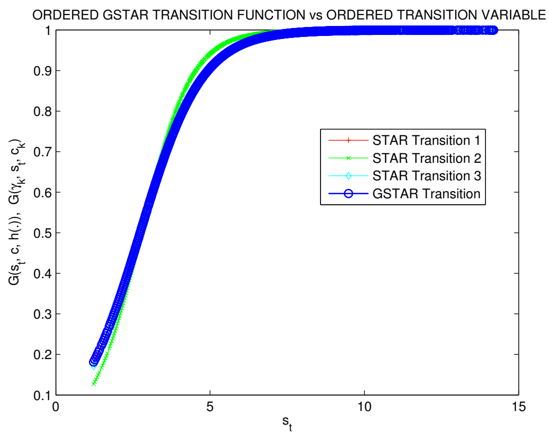
(b) logMSSN: G(Ξ) vs s_t



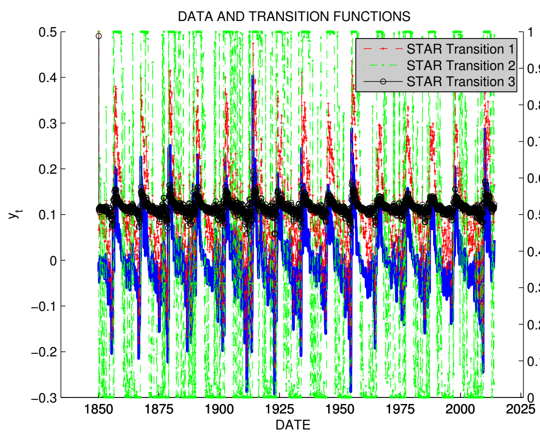
(c) sqrtMSSN: G(Ξ) vs t



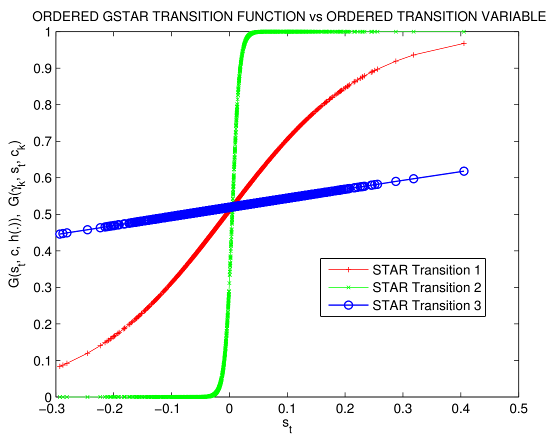
(d) sqrtMSSN: G(Ξ) vs s



(e) DLMSSN: G(Ξ) vs t



(f) DLMSSN: G(Ξ) vs s



NOTE: The data are plotted in blue line of left-had subfigures.

Table 1: Expected values of $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{c}$ resulting from a simulation of the model (1) with $\phi_0 = 0$, $\phi_1 = 0.65$, $\phi_2 = -0.5$, $\theta_0 = 0.02$, $\theta_1 = -0.8$, $\theta_2 = 0.4$, $c = \bar{y}_t$ and various couples of parameters (γ_1, γ_2) for 1000 draws.

	$\gamma_1 = 4$, $\gamma_2 = -1$		$\gamma_1 = 40$, $\gamma_2 = -10$		$\gamma_1 = \gamma_2 = 2$		$\gamma_1 = \gamma_2 = 20$	
T	$E(\hat{\gamma}_1)$	$E(\hat{\gamma}_2)$	$E(\hat{\gamma}_1)$	$E(\hat{\gamma}_2)$	$E(\hat{\gamma}_1)$	$E(\hat{\gamma}_2)$	$E(\hat{\gamma}_1)$	$E(\hat{\gamma}_2)$
25	3.7800	-0.8868	34.9740	-8.2496	1.6303	1.6303	19.0369	19.0369
50	3.8146	-0.8843	35.8474	-8.4687	1.7495	1.7495	19.5855	19.5855
75	3.8211	-0.9033	36.2491	-8.4935	1.7420	1.7420	19.7254	19.7254
100	3.8367	-0.9596	36.8992	-8.8842	1.8292	1.8292	19.9814	19.9814
–								
150	3.9462	-0.9925	37.5510	-9.2111	1.8606	1.8606	20.5156	20.5156
–								
200	3.9481	-1.0428	38.6911	-9.9711	1.8648	1.8648	21.2614	21.2614
–								
300	3.9617	-1.0449	39.6556	-10.2743	1.8814	1.8814	22.4290	22.4290

Table 2: Empirical Size and Power of "All-in-One" and "Two-Step" test for dynamic asymmetry under DGP 1.

			Empirical Size								
T	γ_1	γ_2	LM_1			LM_2			S_1		
			$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
50	0	0	0.0035	0.0110	0.0289	0.0039	0.0128	0.0237	0.0062	0.0452	0.0604
100	0	0	0.0003	0.0221	0.0458	0.0000	0.0230	0.0467	0.0025	0.0135	0.0430
300	0	0	0.0058	0.0255	0.0565	0.0070	0.0288	0.0653	0.0039	0.0168	0.0433
1000	0	0	0.0073	0.0384	0.0798	0.0208	0.0735	0.1297	0.0088	0.0527	0.0882

			Empirical Power								
T	γ_1	γ_2	LM_1			LM_2			S_1		
			$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
	50	10	0.0444	0.1263	0.2315	0.0065	0.0361	0.0587	0.0166	0.0514	0.0939
	-50	10	0.0065	0.0361	0.0587	0.0072	0.0213	0.0563	0.0166	0.0514	0.0939
	50	-10	0.0072	0.0213	0.0563	0.0062	0.0218	0.0555	0.0041	0.0380	0.0822
	50	0	0.0389	0.1215	0.2075	0.0035	0.0050	0.0059	0.0007	0.0182	0.0430
	-50	0	0.0309	0.1149	0.2145	0.0067	0.0312	0.0647	0.0027	0.0304	0.0575
	0	50	0.0391	0.1308	0.2261	0.0065	0.0246	0.0569	0.0003	0.0333	0.0540
50	500	100	0.5364	0.5665	0.6057	0.5158	0.5318	0.5363	0.5193	0.5433	0.5658
	-500	100	0.0364	0.1180	0.2173	0.0065	0.0198	0.0401	0.0113	0.0595	0.0949
	500	-100	0.5355	0.5613	0.5930	0.5158	0.5194	0.5361	0.5196	0.5351	0.5551
	500	0	0.5358	0.5626	0.5967	0.5158	0.5204	0.5369	0.5146	0.5222	0.5392
	-500	0	0.0333	0.1149	0.2154	0.0067	0.0303	0.0655	0.0027	0.0340	0.0618
	0	500	0.5298	0.5801	0.6316	0.5164	0.5221	0.5375	0.5160	0.5256	0.5361
	50	10	0.0315	0.1382	0.2366	0.0142	0.0951	0.1561	0.0271	0.0646	0.1006
	-50	10	0.0197	0.0737	0.1502	0.0054	0.0668	0.1246	0.0231	0.0694	0.1093
	50	-10	0.0180	0.0939	0.1904	0.0100	0.0747	0.1405	0.0175	0.0554	0.0790
	50	0	0.0170	0.0836	0.1601	0.0169	0.0731	0.1448	0.0006	0.0252	0.0443
	-50	0	0.0094	0.0801	0.1288	0.0164	0.0767	0.1364	0.0072	0.0416	0.0765
	0	50	0.0128	0.0732	0.1486	0.0128	0.0732	0.1486	0.0034	0.0226	0.0491
100	500	100	0.7930	0.8165	0.8335	0.7927	0.7987	0.8191	0.7986	0.8049	0.8130
	-500	100	0.0173	0.0890	0.1645	0.0079	0.0760	0.1238	0.0215	0.0640	0.1173
	500	-100	0.7931	0.8073	0.8241	0.7927	0.8031	0.8155	0.7965	0.8004	0.8259
	500	0	0.7954	0.8059	0.8215	0.7947	0.8067	0.8189	0.7928	0.7959	0.8004
	-500	0	0.0094	0.0782	0.1325	0.0164	0.0776	0.1394	0.0114	0.0466	0.0795
	0	500	0.8037	0.8083	0.8303	0.8037	0.8083	0.8303	0.8012	0.8083	0.8165
	50	10	0.1098	0.2570	0.3719	0.0281	0.1295	0.2018	0.0681	0.1501	0.2173
	-50	10	0.0207	0.1266	0.1941	0.0102	0.0640	0.1392	0.0390	0.0946	0.1496
	50	-10	0.0420	0.1458	0.2253	0.0132	0.0729	0.1561	0.0287	0.0890	0.1509
	50	0	0.0353	0.1339	0.2059	0.0212	0.0823	0.1496	0.0103	0.0363	0.0634
	-50	0	0.0190	0.1033	0.1476	0.0323	0.1179	0.2085	0.0117	0.0402	0.0697
	0	50	0.0176	0.0964	0.1898	0.0176	0.0964	0.1898	0.0092	0.0359	0.0729
300	500	100	0.9958	0.9961	0.9980	0.9958	0.9958	0.9958	0.9959	0.9999	1.0000
	-500	100	0.0360	0.1485	0.2147	0.0096	0.0765	0.1389	0.0468	0.1197	0.1856
	500	-100	0.9958	0.9961	0.9961	0.9958	0.9961	0.9961	1.0000	1.0000	1.0000
	500	0	1.0000	1.0000	1.0000	0.9958	0.9961	0.9961	1.0000	1.0000	1.0000
	-500	0	0.0190	0.1039	0.1434	0.0326	0.1275	0.2074	0.0122	0.0428	0.0702
	0	500	0.9917	0.9966	0.9972	0.9917	0.9966	0.9972	1.0000	1.0000	1.0000
	50	10	0.4727	0.7274	0.8419	0.1173	0.2498	0.3634	0.2139	0.3545	0.4518
	-50	10	0.0830	0.2032	0.2996	0.0274	0.0822	0.1506	0.0580	0.1589	0.2205
	50	-10	0.1411	0.2863	0.4373	0.0312	0.0942	0.1778	0.0866	0.2054	0.2599
	50	0	0.1344	0.3275	0.4455	0.0476	0.1435	0.2856	0.0107	0.0635	0.1281
	-50	0	0.0446	0.1193	0.2482	0.0846	0.2535	0.3520	0.0110	0.0471	0.0893
	0	50	0.0654	0.1799	0.2643	0.0654	0.1799	0.2643	0.0116	0.0586	0.1325
1000	500	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	-500	100	0.1281	0.2695	0.4265	0.0242	0.0719	0.1607	1.0000	1.0000	1.0000
	500	-100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.1181	0.2180	0.2752
	500	0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	-500	0	0.0462	0.1223	0.2390	0.0890	0.2565	0.3556	1.0000	1.0000	1.0000
	0	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3: Empirical Size and Power of "All-in-One" and "Two-Step" test for dynamic asymmetry under DGP 2.

			Empirical Size								
T	γ_1	γ_2	LM_1			LM_2			S_1		
			$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
50	0	0	0.0008	0.0031	0.0048	0.0026	0.0154	0.0277	0.0048	0.0136	0.0289
100	0	0	0.0024	0.0164	0.0185	0.0014	0.0131	0.0237	0.0006	0.0264	0.0423
300	0	0	0.0000	0.0081	0.0272	0.0002	0.0123	0.0242	0.0112	0.0292	0.0469
1000	0	0	0.0036	0.0460	0.1018	0.0015	0.0391	0.0787	0.0126	0.0482	0.0863

			Empirical Power								
T	γ_1	γ_2	LM_1			LM_2			S_1		
			$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
	50	10	0.0265	0.1037	0.1549	0.0048	0.0322	0.0570	0.0050	0.0205	0.0476
	-50	10	0.0302	0.0938	0.1522	0.0048	0.0305	0.0564	0.0109	0.0415	0.0588
	50	-10	0.0284	0.0972	0.1778	0.0048	0.0054	0.0386	0.0658	0.0225	0.0510
	50	0	0.0259	0.0961	0.1694	0.0054	0.0352	0.0593	0.0048	0.0156	0.0405
	-50	0	0.0277	0.0869	0.1711	0.0052	0.0258	0.0613	0.0103	0.0321	0.0553
	0	50	0.0052	0.0258	0.0613	0.0079	0.0320	0.0548	0.0048	0.0169	0.0340
50	500	100	0.0273	0.0810	0.1468	0.0316	0.1009	0.1760	0.0009	0.0237	0.0435
	-500	100	0.0277	0.0951	0.1725	0.0052	0.0361	0.0552	0.0048	0.0162	0.0337
	500	-100	0.0316	0.1009	0.1760	0.0090	0.0341	0.0636	0.0059	0.0226	0.0572
	500	0	0.0277	0.0926	0.1714	0.0052	0.0307	0.0648	0.0048	0.0158	0.0289
	-500	0	0.0277	0.0951	0.1725	0.0052	0.0361	0.0552	0.0103	0.0367	0.0491
	0	500	0.0277	0.0926	0.1714	0.0052	0.0307	0.0648	0.0048	0.0142	0.0289
	50	10	0.0308	0.1249	0.2454	0.0111	0.0328	0.1117	0.0031	0.0261	0.0585
	-50	10	0.0185	0.1021	0.1847	0.0117	0.0262	0.0520	0.0100	0.0393	0.0698
	50	-10	0.0308	0.1249	0.2454	0.0111	0.0328	0.1117	0.0031	0.0411	0.0652
	50	0	0.0308	0.1249	0.2454	0.0111	0.0328	0.1117	1.0000	1.0000	1.0000
	-50	0	0.0185	0.1021	0.1847	0.0111	0.0262	0.0520	0.0098	0.0354	0.0697
	0	50	0.0185	0.0930	0.1705	0.0108	0.0262	0.0560	1.0000	1.0000	1.0000
100	500	100	0.0192	0.0950	0.1893	0.0108	0.0247	0.0544	0.0006	0.0159	0.0416
	-500	100	0.0189	0.1028	0.1786	0.0117	0.0262	0.0520	0.0111	0.0485	0.0800
	500	-100	0.0192	0.0950	0.1893	0.0108	0.0247	0.0544	0.0044	0.0349	0.0574
	500	0	0.0192	0.0950	0.1893	0.0108	0.0247	0.0544	1.0000	1.0000	1.0000
	-500	0	0.0189	0.1028	0.1786	0.0117	0.0262	0.0520	1.0000	1.0000	1.0000
	0	500	0.0185	0.0930	0.1705	0.0108	0.0262	0.0560	1.0000	1.0000	1.0000
	50	10	0.1044	0.2473	0.3516	0.0510	0.1327	0.2083	0.0099	0.0230	0.0409
	-50	10	0.0221	0.0957	0.1983	0.0103	0.0398	0.0681	0.0141	0.0315	0.0757
	50	-10	0.1044	0.2473	0.3516	0.0510	0.1327	0.2083	0.0138	0.0258	0.0511
	50	0	0.1044	0.2473	0.3516	0.0510	0.1327	0.2083	0.0099	0.0230	0.0387
	-50	0	0.0221	0.0957	0.1983	0.0103	0.0398	0.0681	0.0141	0.0305	0.0757
	0	50	0.0190	0.0929	0.1907	0.0084	0.0379	0.0666	0.0112	0.0292	0.0478
300	500	100	0.0329	0.1381	0.2106	0.0165	0.0575	0.0906	0.0131	0.0189	0.0369
	-500	100	0.0221	0.0957	0.1953	0.0103	0.0394	0.0623	0.0167	0.0443	0.0820
	500	-100	0.0329	0.1381	0.2106	0.0165	0.0575	0.0906	0.0131	0.0252	0.0533
	500	0	0.0329	0.1381	0.2106	0.0165	0.0575	0.0906	1.0000	1.0000	1.0000
	-500	0	0.0221	0.0957	0.1953	0.0103	0.0394	0.0623	1.0000	1.0000	1.0000
	0	500	0.0190	0.0929	0.1907	0.0084	0.0379	0.0666	1.0000	1.0000	1.0000
	50	10	0.5292	0.7540	0.8404	0.3287	0.5776	0.6915	0.0082	0.0251	0.0541
	-50	10	0.0964	0.2863	0.4173	0.0240	0.1301	0.2150	0.0073	0.0370	0.0527
	50	-10	0.5292	0.7540	0.8404	0.3287	0.5776	0.6915	0.0087	0.0322	0.0686
	50	0	0.5292	0.7540	0.8404	0.3287	0.5776	0.6915	1.0000	1.0000	1.0000
	-50	0	0.0964	0.2863	0.4173	0.0240	0.1301	0.2150	1.0000	1.0000	1.0000
	0	50	0.0888	0.2829	0.4009	0.0208	0.1199	0.2118	1.0000	1.0000	1.0000
1000	500	100	0.1872	0.4034	0.5708	0.0693	0.2030	0.3046	0.0020	0.0181	0.0379
	-500	100	0.0926	0.2848	0.4135	0.0233	0.1267	0.2122	0.0087	0.0385	0.0656
	500	-100	0.1872	0.4034	0.5708	0.0693	0.2030	0.3046	0.0057	0.0290	0.0533
	500	0	0.1872	0.4034	0.5708	0.0693	0.2030	0.3046	1.0000	1.0000	1.0000
	-500	0	0.0926	0.2848	0.4135	0.0233	0.1267	0.2122	1.0000	1.0000	1.0000
	0	500	0.0888	0.2829	0.4009	0.0208	0.1199	0.2118	1.0000	1.0000	1.0000

Table 4: Empirical Size and Empirical Power of test for serial correlation, no additive asymmetry and parameter constancy under DGP 1.

Empirical Size

T	γ_1	γ_2	γ_3	γ_4	Nominal size	No error autocorrelation $\rho = 0$				No additional asymmetry	Parameter constancy		
						q=1	q=2	q=4	q=10	H_0	LM_1	LM_2	LM_3
100	2	1			$\alpha = 0.01$	0.0060	0.0041	0.0087	0.0053	0.0020	0.0000	0.0000	0.0005
					$\alpha = 0.05$	0.0628	0.0482	0.0354	0.0435	0.0034	0.0000	0.0000	0.0005
					$\alpha = 0.10$	0.0991	0.1008	0.0931	0.0995	0.0063	0.0000	0.0000	0.0029
	20	10			$\alpha = 0.01$	0.0045	0.0134	0.0070	0.0040	0.0000	0.0000	0.0000	0.0000
					$\alpha = 0.05$	0.0526	0.0392	0.0186	0.0167	0.0007	0.0000	0.0000	0.0023
					$\alpha = 0.10$	0.0970	0.0953	0.0515	0.0393	0.0020	0.0000	0.0000	0.0092
	200	100			$\alpha = 0.01$	0.0102	0.0149	0.0098	0.0047	0.0000	0.0000	0.0000	0.0000
					$\alpha = 0.05$	0.0459	0.0478	0.0389	0.0318	0.0000	0.0000	0.0000	0.0012
					$\alpha = 0.10$	0.0921	0.1024	0.1008	0.0491	0.0080	0.0000	0.0012	0.0040
300	2	1			$\alpha = 0.01$	0.0136	0.0114	0.0112	0.0057	0.0024	0.0000	0.0000	0.0000
					$\alpha = 0.05$	0.0462	0.0527	0.0616	0.0436	0.0065	0.0000	0.0000	0.0023
					$\alpha = 0.10$	0.1078	0.0969	0.1075	0.0983	0.0122	0.0000	0.0000	0.0081
	20	10			$\alpha = 0.01$	0.0087	0.0084	0.0043	0.0014	0.0018	0.0000	0.0000	0.0041
					$\alpha = 0.05$	0.0622	0.0563	0.0287	0.0112	0.0018	0.0000	0.0000	0.0297
					$\alpha = 0.10$	0.1245	0.1210	0.0708	0.0241	0.0099	0.0000	0.0023	0.0607
	200	100			$\alpha = 0.01$	0.0250	0.0137	0.0149	0.0000	0.0142	0.0000	0.0000	0.0045
					$\alpha = 0.05$	0.0621	0.0043	0.0618	0.0208	0.0657	0.0000	0.0023	0.0344
					$\alpha = 0.10$	0.1221	0.0124	0.1195	0.0356	0.1083	0.0000	0.0057	0.0836

Empirical Power

T	γ_1	γ_2	γ_3	γ_4	Nominal size	No error autocorrelation $\rho = 0.2$				No error autocorrelation $\rho = 0.4$				No additional asymmetry	Parameter constancy		
						q=1	q=2	q=4	q=10	q=1	q=2	q=4	q=10	H_1	LM_1	LM_2	LM_3
100	2	1	5	2	$\alpha = 0.01$	0.0075	0.0074	0.1261	0.6185	0.0276	0.0295	0.8493	0.9916	0.1744	0.2313	0.3280	0.5316
					$\alpha = 0.05$	0.0520	0.0594	0.3388	0.7702	0.0840	0.1011	0.9535	1.0000	0.2157	0.2957	0.4577	0.6858
					$\alpha = 0.10$	0.1111	0.0952	0.4723	0.8719	0.1530	0.1897	0.9706	1.0000	0.2516	0.3333	0.5354	0.7378
	20	10	50	20	$\alpha = 0.01$	0.0112	0.0138	0.0863	0.5425	0.0227	0.0228	0.7557	0.9930	0.1747	0.2270	0.3257	0.5356
					$\alpha = 0.05$	0.0589	0.0621	0.2024	0.7090	0.0777	0.1014	0.9013	0.9959	0.2161	0.2768	0.4403	0.6958
					$\alpha = 0.10$	0.1033	0.1007	0.3262	0.7988	0.1371	0.1834	0.9465	1.0000	0.2496	0.3228	0.5361	0.7517
	200	100	500	200	$\alpha = 0.01$	0.0075	0.0203	0.0934	0.5540	0.0158	0.0290	0.7552	0.9931	1.0000	0.7871	0.8489	0.8924
					$\alpha = 0.05$	0.0611	0.1011	0.2538	0.7341	0.0759	1.0000	0.9035	1.0000	1.0000	0.8239	0.8770	0.9230
					$\alpha = 0.10$	0.1027	0.1146	0.3696	0.8176	0.1376	0.1767	0.9490	1.0000	1.0000	0.8296	0.8934	0.9331
300	2	1	5	2	$\alpha = 0.01$	0.0290	0.0312	0.8014	1.0000	0.1047	0.1235	1.0000	1.0000	0.0118	0.0215	0.3353	0.7728
					$\alpha = 0.05$	0.0940	0.0953	0.9070	1.0000	0.2974	0.2724	1.0000	1.0000	0.0535	0.0835	0.6957	0.9442
					$\alpha = 0.10$	0.1859	0.1914	0.9467	1.0000	0.4006	0.4229	1.0000	1.0000	0.1149	0.1804	0.8281	0.9917
	20	10	50	20	$\alpha = 0.01$	0.0987	0.0810	0.7414	0.9968	0.0890	0.1331	1.0000	1.0000	0.1747	0.0215	0.3353	0.7728
					$\alpha = 0.05$	0.1866	0.1652	0.8827	0.9980	0.2430	0.2848	1.0000	1.0000	0.2161	0.0835	0.6957	0.9442
					$\alpha = 0.10$	0.2372	0.2323	0.9069	1.0000	0.3436	0.4272	1.0000	1.0000	0.2496	0.1804	0.8281	0.9940
	200	100	500	200	$\alpha = 0.01$	0.0216	0.0419	0.7046	0.9968	0.0860	0.1422	1.0000	1.0000	1.0000	0.9961	0.9961	0.9961
					$\alpha = 0.05$	0.1106	0.1110	0.8476	1.0000	0.2156	0.2980	1.0000	1.0000	1.0000	0.9961	0.9970	1.0000
					$\alpha = 0.10$	0.1755	0.1986	0.8963	1.0000	0.3349	0.4364	1.0000	1.0000	1.0000	0.9968	0.9991	1.0000

Table 5: Empirical Size and Empirical Power of test for serial correlation, no additive asymmetry and parameter constancy under DGP 2.

Empirical Size

T	γ_1	γ_2	γ_3	γ_4	Nominal size	No error autocorrelation $\rho = 0$ q=1	q=2	q=4	q=10	No additional asymmetry H_0	Parameter constancy LM_1	LM_2	LM_3
					$\alpha = 0.01$	0.0082	0.0064	0.0107	0.0039	0.0055	0.0000	0.0000	0.0000
	2	1			$\alpha = 0.05$	0.0534	0.0453	0.0503	0.0443	0.0464	0.0000	0.0000	0.0000
					$\alpha = 0.10$	0.1120	0.1124	0.1119	0.0781	0.0945	0.0000	0.0000	0.0000
					$\alpha = 0.01$	0.0059	0.0069	0.0074	0.0029	0.0011	0.0000	0.0000	0.0000
100	20	10			$\alpha = 0.05$	0.0488	0.0421	0.0400	0.0333	0.0425	0.0000	0.0000	0.0041
					$\alpha = 0.10$	0.0978	0.0948	0.1062	0.0819	0.0812	0.0000	0.0000	0.0053
					$\alpha = 0.01$	0.0084	0.0059	0.0046	0.0018	0.0345	0.0000	0.0000	0.0000
	200	100			$\alpha = 0.05$	0.0433	0.0470	0.0446	0.0163	0.0700	0.0000	0.0000	0.0000
					$\alpha = 0.10$	0.0983	0.1089	0.0977	0.0455	0.1089	0.0000	0.0000	0.0005
					$\alpha = 0.01$	0.0096	0.0098	0.0159	0.0058	0.0118	0.0000	0.0000	0.0000
	2	1			$\alpha = 0.05$	0.0448	0.0479	0.0506	0.0532	0.0512	0.0000	0.0000	0.0000
					$\alpha = 0.10$	0.0899	0.0890	0.1006	0.0926	0.0925	0.0000	0.0000	0.0000
					$\alpha = 0.01$	0.0099	0.0083	0.0164	0.0051	0.0127	0.0000	0.0000	0.0000
300	20	10			$\alpha = 0.05$	0.0463	0.0446	0.0484	0.0461	0.0505	0.0000	0.0000	0.0048
					$\alpha = 0.10$	0.0884	0.0920	0.0882	0.0905	0.0916	0.0000	0.0000	0.0052
					$\alpha = 0.01$	0.0128	0.0134	0.0111	0.0100	0.0381	0.0000	0.0000	0.0000
	200	100			$\alpha = 0.05$	0.0584	0.0413	0.0611	0.0481	0.0640	0.0000	0.0000	0.0000
	(120)	(70)			$\alpha = 0.10$	0.0891	0.0853	0.1013	0.0968	0.1217	0.0000	0.0000	0.0000

Empirical Power

T	γ_1	γ_2	γ_3	γ_4	Nominal size	No error autocorrelation $\rho = 0.2$ q=1	q=2	q=4	q=10	No error autocorrelation $\rho = 0.4$ q=1	q=2	q=4	q=10	No additional asymmetry H_1	Parameter constancy LM_1	LM_2	LM_3
					$\alpha = 0.01$	0.0538	0.0819	0.2704	0.6581	0.3615	0.4917	0.9581	0.9892	0.1299	0.0019	0.0042	0.0277
	2	1	5	2	$\alpha = 0.05$	0.1772	0.2302	0.5272	0.8088	0.5759	0.7035	0.9842	0.9919	0.1675	0.0075	0.0456	0.1205
					$\alpha = 0.10$	0.2873	0.3440	0.6532	0.8812	0.7180	0.7885	0.9881	0.9959	0.2029	0.0197	0.0880	0.1992
					$\alpha = 0.01$	0.0540	0.0633	0.2863	0.6560	0.3746	0.4811	0.9618	0.9892	0.1299	0.0016	0.0186	0.1731
100	20	10	50	20	$\alpha = 0.05$	0.1817	0.2241	0.5427	0.8127	0.6207	0.7083	0.9858	0.9960	0.1675	0.0121	0.0807	0.4039
					$\alpha = 0.10$	0.2749	0.3312	0.6689	0.8792	0.7587	0.8168	0.9881	0.9960	0.2029	0.0223	0.1597	0.5689
					$\alpha = 0.01$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0059	0.0566	0.1340
	200	100	500	200	$\alpha = 0.05$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0207	0.1591	0.3303
					$\alpha = 0.10$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0503	0.2475	0.4782
					$\alpha = 0.01$	0.3156	0.3919	0.9576	1.0000	0.9431	0.9979	1.0000	1.0000	0.3567	0.0000	0.0015	0.0649
	2	1	5	2	$\alpha = 0.05$	0.5697	0.6956	0.9806	1.0000	0.9873	0.9997	1.0000	1.0000	0.3777	0.0000	0.0225	0.2348
					$\alpha = 0.10$	0.7064	0.7885	0.9890	1.0000	0.9948	1.0000	1.0000	1.0000	0.4125	0.0001	0.0509	0.3599
					$\alpha = 0.01$	0.3455	0.4193	0.9622	1.0000	0.9565	0.9983	1.0000	1.0000	0.3567	0.0000	0.0015	0.0609
300	20	10	50	20	$\alpha = 0.05$	0.6141	0.7093	0.9819	1.0000	0.9921	1.0000	1.0000	1.0000	0.3777	0.0000	0.0225	0.2349
					$\alpha = 0.10$	0.7394	0.8015	0.9922	1.0000	0.9964	1.0000	1.0000	1.0000	0.4125	0.0001	0.0509	0.3599
					$\alpha = 0.01$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0015	0.0649
	200	100	500	200	$\alpha = 0.05$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0225	0.2348
					$\alpha = 0.10$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0001	0.0509	0.3599

Table 6: Datasets

Series	Sample	T^s	Testing Period	Source	Previous Studies
IP	1947 M—2013 M3	500	1982 M9+h — 2013M3-12+h	OECD Main Economic Indicators (2007=100) Real Data (Release: March 2013)	Anderson and Teräsvirta (1992) Proietti (1998) Teräsvirta et al. (2005)
UN	1948 M—2013 M3	500	1983 M9+h — 2013M3-12+h	OECD Main Economic Indicators (2007=100) Real Data (Release: March 2013)	Rothman (1991) Rothman (1998) Montgomery et al. (1998) Skalin and Teräsvirta (2002) Proietti (2003)
YSSN	1700—2013	150	1851+h — 2013-12+h	Solar Influences Data Analysis Center *Yearly mean total sunspot number* http://sidc.oma.be/silso/infosnytot	Tong and Lim (1980) Tsay (1989) Hansen (1999) Strikholm and Teräsvirta (2005)
LYNX	1820—1934	50	1871+h — 1934-12+h	The Encyclopedia of Mathematics wiki www.encyclopediaofmath.org/index.php/Canadian_lynx_series	Elton and Nicholson (1942) Moran (1953) Tong (1977) Tsay (1989) Teräsvirta (1994)
ENT	1977:Q1—2013:Q4	50	1989:Q3+h — 2013:Q4-4+h	https://sites.google.com/site/ezcwebsite/archium---data	Rossi and Zanetti Chini (2016)
EXT	1977:Q1—2013:Q4	50	1989:Q3+h — 2013:Q4-4+h	https://sites.google.com/site/ezcwebsite/archium---data	Rossi and Zanetti Chini (2016)

Table 7: Empirical application of the (G)STAR model to real data samples

Descriptive statistics

Series	Mean	Median	MAD	Std.Dev.	Skewness	Kurtosis	JB[a]	ARCH-effects[b]	DW[c]	KS[d]	ADF[(1)(3)]
IP	0.2490	0.2790	0.3474	0.4780	-0.2815	4.5233	0.0010	0.0000	0.0293	0.0000	-11.1962**
UN	0.0974	-0.3704	1.3182	1.8054	0.9802	4.9645	0.0010	0.0000	0.9341	0.0001	-10.7775**
YSSN	6.3981	6.3246	2.4452	2.9486	0.1790	2.2904	0.0226	0.0000	0.2978	0.0000	-5.2865**
LYNX	2.9037	2.8870	0.4721	0.5584	-0.3620	2.2664	0.0582	0.0024	0.0176	0.0000	-5.8245**

Linearity and Asymmetry tests (*p*-values)

Series	Linearity Test[(2)]				Model[(2)]	Symmetry Test	
	F_L	F_3	F_2	F_1		All-in-One	Two-Step
IP	0.053	0.016	0.223	0.560	LSTAR1	0.0150	1.0000
UN	$2.1e^{-5}$	0.016	0.569	$1.6\ e^{-5}$	LSTAR1	0.0005	.9900
YSSN	$3.03e^{-5}$	0.140	0.940	$2.99e^{-7}$	LSTAR1	0.0209	0.9955
LYNX	0.002	$1.6\ e^{-4}$	0.614	0.166	LSTAR1	0.0001	1.0000

Estimates

Parameter	(MR)STAR IP Value	(MR)STAR IP SE	(MR)STAR UN Value	(MR)STAR UN SE	(MR)STAR YSSN Value	(MR)STAR YSSN SE	(MR)STAR LYNX Value	(MR)STAR LYNX SE	GSTAR IP Value	GSTAR IP SE	GSTAR UN Value	GSTAR UN SE	GSTAR YSSN Value	GSTAR YSSN SE	GSTAR LYNX Value	GSTAR LYNX SE
ϕ_0	0.0007	0.0105	-0.0217	0.0319	0.8386	0.4665	3.6129	8.9173	-0.0317	0.0304	-0.1318	0.1212	0.0016	0.8946	3.0571	3.5488
ϕ_1	1.2982	0.0534	0.8624	0.0511	1.3673	0.1859	-0.2229	1.2204	1.2509	0.0809	0.8283	0.0856	0.0017	0.5732	-0.8190	3.7252
ϕ_2	-0.2449	0.0846	0.2670	0.0659	-0.6531	0.2167	0.2898	0.7223	-0.1757	0.1228	0.2150	0.0903	-0.0013	1.0925	0.9297	3.0689
ϕ_3	0.0303	0.0837	-0.2039	0.0490	-0.0246	0.2418	0.0388	0.1648	0.0389	0.1263	-0.1499	0.0751	0.0001	0.5034	-0.1618	1.0542
ϕ_4	-0.0289	0.0821	-	-	-0.0596	0.2221	2.5471	6.5730	0.0422	0.1323	-	-	-0.0001	0.3340	0.6786	1.9507
ϕ_5	-0.0671	0.0818	-	-	0.2429	0.1534	0.1100	0.9837	-0.1608	0.1318	-	-	0.0002	0.3513	0.3750	0.7873
ϕ_6	-0.1055	0.0555	-	-	-	-	-0.9993	1.4859	-0.1499	0.0835	-	-	-	-	-0.9116	1.8615
ϕ_7	-	-	-	-	-	-	-0.0281	0.5210		-	-	-	-	-	0.2160	0.4393
θ_{10}	0.0626	0.0320	0.0805	0.0582	1.8438	1.1950	-10.7093	41.3355	0.0787	0.0433	0.3241	0.2318	0.7902	1.2649	-2.0815	3.5317
θ_{11}	-0.1975	0.0893	0.3500	0.0725	0.2799	0.2938	2.2348	2.7972	-0.0582	0.1109	0.4011	0.1103	-0.5954	0.5798	2.1757	3.7350
θ_{12}	0.2327	0.1359	-0.2121	0.1058	-0.5836	0.3459	-1.3248	1.6504	0.0566	0.1668	-0.1428	0.1525	1.1665	1.0964	-1.7943	3.1006
θ_{13}	-0.1057	0.1362	-0.1326	0.0708	-0.0460	0.3298	0.1796	0.4185	-0.0944	0.1733	-0.2521	0.1081	-0.5191	0.5365	0.5972	1.1143
θ_{14}	0.0044	0.1390	-	-	0.2602	0.3379	-1.7566	2.7348	-0.1125	0.1839	-	-	0.0003	0.3830	-1.1562	2.0178
θ_{15}	0.1746	0.1411	-	-	-0.3025	0.2201	-0.1107	1.6046	0.0269	0.1806	-	-	-0.0003	0.3680	-0.2925	0.8333
θ_{16}	0.2427	0.0891	-	-	-	-	1.5962	2.8549	0.2337	0.1092	-	-	-	-	1.0047	1.8314
θ_{17}	-	-	-	-	-	-	0.1031	0.7856		-	-	-	-	-	-0.1760	0.4510
θ_{20}	-	-	-	-	-0.3877	1.0656	0.5321	3.5232	-	-	-	-	-	-	-	-
θ_{21}	-	-	-	-	-0.5768	0.2432	-0.7867	0.7169	-	-	-	-	-	-	-	-
θ_{22}	-	-	-	-	1.1635	0.3362	-1.3103	0.7344	-	-	-	-	-	-	-	-
θ_{23}	-	-	-	-	-0.3871	0.2829	3.0855	1.2053	-	-	-	-	-	-	-	-
θ_{24}	-	-	-	-	-0.0173	0.3069	-2.8533	1.4792	-	-	-	-	-	-	-	-
θ_{25}	-	-	-	-	-0.0099	0.1914	3.9498	1.7185	-	-	-	-	-	-	-	-
θ_{26}	-	-	-	-	-	-	-2.8365	1.6553	-	-	-	-	-	-	-	-
θ_{27}	-	-	-	-	-	-	0.1947	0.5976								
γ_1	4.7500	0.0320	29.0890	0.0582	129.56	1.1950	0.7807	41.3355	5.0000	0.0433	0.0001	0.2318	3.3171	1.2649	-15.000	0.202
γ_2	-	-	-		5.9	0.0010	9.8119	46.587	3.0000	0.1109	0.0001	0.1103	-3.3171	0.5798	17.000	0.334
c_1	0.6644	0.0893	0.0590	0.0725	1.2482	0.2938	2.6201	2.7972	-0.0446	0.1668	0.2900	0.1525	6.3746	1.0964	2.3006	1.475
c_2	-	-	-	-	2.0760	0.0001	4.6587	-	-	-	-	-	-	-	-	-

Diagnostics (*p*-values)

Diagnostic Test	STAR IP	STAR UN	STAR YSSN	STAR LYNX	GSTAR IP	GSTAR UN	GSTAR YSSN	GSTAR LYNX
No error autocorrelation								
q=1	0.1785	0.1014	0.3341	0.2398	0.3336	0.1434	0.2411	0.6612
q=2	0.4028	0.2616	0.6278	0.5018	0.5507	0.3430	0.5036	0.9089
q=4	0.0001	0.6130	0.9206	0.8486	0.0086	0.7106	0.8499	0.9958
q=10	0.0000	0.9882	0.9999	0.9993	0.0000	0.9953	0.9993	1.0000
No rem. nonlinearity / No rem. asymmetry	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Parameter constancy								
H1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
H2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
H3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

NOTE: (a) Jarque-Brera ($p-$value) (b) Engle's test ($p-$value); (c) Durbin-Watson test ($p-$value); (d) Kolmogorov-Smirnov test ($p-$value); (1)Values expressed in test-statistics, significance denoted by '*' (5%), '**' (1%); (2) Results acquired by JMulTi; (3)Results acquired by RATS

Table 8: (Continue) Empirical application of the (G)STAR model to real data samples

Descriptive statistics

Series	Mean	Median	MAD	Std.Dev.	Skewness	Kurtosis	JB[a]	ARCH-effects[b]	DW[c]	KS[d]	ADF[(1)]
ENT	0.0329	0.2369	1.6581	2.1785	-0.1508	3.1368	0.5000	0.0013	0.0075	0.0002	-5.7565**
EXT	0.1290	0.5156	1.6354	2.1771	-0.8640	3.7131	0.0029	0.0008	0.9466	0.0004	-4.2367**

Linearity and Asymmetry tests (p-values)[(2)]

Series	Linearity Test					Symmetry Test	
	F_L	F_3	F_2	F_1	Model	All-in-One	Two-Step
ENT	0.6758	0.4953	0.7389	0.7389	Linear	0.9210	0.9725
EXT	0.0060	0.0522	0.0785	0.0351	(Symmetric) LSTAR1	0.0336	1.0000

Estimates

Parameter	(MR)STAR				GSTAR			
	ENT		EXT		ENT		EXT	
	Value	SE	Value	SE	Value	SE	Value	SE
ϕ_0	-0.0278	0.0766	1.0979	0.8782	-0.0991	0.1134	0.9832	0.7164
ϕ_1	1.6883	0.0987	1.4693	0.2724	1.6879	0.1101	1.4286	0.2421
ϕ_2	-1.0976	0.1544	-0.3538	0.2189	-1.1293	0.1654	-0.3289	0.2253
ϕ_3	0.1953	0.0907	-0.3952	0.1471	0.1988	0.0992	-0.4024	0.1556
θ_{10}	0.6075	4.0147	0.3538	1.6924	1.5726	2.0553	-1.0953	0.7233
θ_{11}	4.6813	3.4876	0.7729	1.6938	-0.4129	0.5271	0.2291	0.2694
θ_{12}	-8.1335	3.2898	0.3397	0.8342	0.2468	0.4207	-0.6216	0.2830
θ_{13}	4.0331	1.9560	0.1719	0.5085	-0.2725	0.2914	0.5309	0.1859
θ_{20}	8.6009	4.0147	-1.6048	1.8228	.	.	.	.
θ_{21}	280.2337	436.2670	-0.3819	1.8350	.	.	.	.
θ_{22}	0.0067	1.9410	-1.1769	8.0053	.	.	.	.
θ_{23}	0.0013	1.9645	0.4003	5.0216	.	.	.	.
γ_1	1.6009	1.4876	7.9000	11.9000	0.0001	2.0553	3.2500	0.7233
γ_2	0.2337	3.5218	101.3000	590.5800	0.2500	0.5271	3.2500	0.2694
c_1	0.6833	4.0147	-0.7730	3.0000	2.4648	0.4207	-1.8588	0.2830
c_2	0.7441	3.4876	-0.2768	0.9620	.	.	.	.

Diagnostics (p-values)

Diagnostic Test	STAR		GSTAR	
No error autocorrelation				
q=1	0.0523	0.3325	0.0000	0.0920
q=2	0.0115	0.6186	0.0002	0.1850
q=4	0.0000	0.0000	0.0000	0.0000
q=10	0.0001	0.0000	0.0000	0.0000
No rem. nonlinearity / No rem. asymmetry	0.9955	1.0000	1.0000	0.9200
Parameter constancy				
H1	0.8256	0.9996	0.9845	0.9993
H2	0.0102	0.9275	0.0919	0.8909
H3	0.0000	0.8400	0.0001	0.7587

NOTE: (a) Jarque-Brera ($p-$value) (b) Engle's test ($p-$value); (c) Durbin-Watson test ($p-$value); (d) Kolmogorov-Smirnov test ($p-$value); (1)Values expressed in test-statistics, significance denoted by '$*$' (5%), '$**$' (1%); (2) Results acquired by RATS

Table 9: Empirical application of the (G)STAR model to Monthly Smoothed Sunspot Number from 1850 to 2013.

Descriptive statistics

Series	Mean	Median	MAD	Std.Dev.	Skewness	Kurtosis	JB[a]	ARCH-effects[b]	DW[c]	KS[d]	ADF[(1)(3)]
logSSN	3.5843	3.8597	0.8641	1.0484	-0.7149	2.7842	0.0010	0.0000	0.0000	0.0000	-4.3619**
sqrtSSN	6.7691	6.8884	2.5196	3.0051	0.1714	2.1951	0.0010	0.0000	0.0000	0.0000	-5.37187**
DLSSN	0.0021	-0.0076	0.0563	0.1224	22.6304	798.4572	0.0010	0.9901	0.0000	0.0000	-36.5383**

Linearity and Asymmetry tests (p-values)

Series	Linearity Test[(2)]					Symmetry Test	
	F_L	F_3	F_2	F_1	Model[(2)]	All-in-One	Two-Step
logSSN	$1.05e^{-35}$	0.0027	$1.20e^{-14}$	$5.92e^{-23}$	LSTAR1	0.0002	1.0000
sqrtSSN	0.0002	0.2410	$5.57e^{-5}$	0.1210	LSTAR2	0.0004	1.0000
DLSSN	0.0012	0.9780	$4.57e^{-4}$	0.0123	LSTAR2	0.0006	1.0000

Estimates

Parameter	(MR)STAR						GSTAR					
	logSSN		sqrtSSN		DLSSN		logSSN		sqrtSSN		DLSSN	
	Value	SE	Value	SE	Value	SE	Value	SE	Value	SE	Value	SE
ϕ_0	-2,495.1000	0.3420	-21.8270	1.4187	-0.3034	0.2572	0.5143	0.0555	1.0461	0.1757	-9.2453	48.1169
ϕ_1	-464.4000	2.1995	-32.574	0.9746	0.1760	0.5590	1.9919	0.1757	1.0461	0.3804	-5.4026	30.5430
ϕ_2	-67.1761	0.7843	-1.412	0.2425	-1.6014	0.8251	-2.0721	0.3724	-2.8801	0.8284	2.5303	10.4673
ϕ_3	31.4922	0.5348	3.1192	0.2704	1.5567	0.9119	1.1495	0.3527	1.6989	0.7672	-2.1641	8.9374
ϕ_4	-16.3030	0.4879	-3.2854	0.2380	-1.0027	0.6947	-0.3047	0.3030	-0.3700	0.6026	1.2448	5.2104
ϕ_5	18.6892	0.9051	0.9386	0.1156	0.7084	0.5401	-0.3126	0.1556	-0.3191	0.2778	-0.7517	3.2121
ϕ_6	-	-	-	-	-0.3175	0.3528	-	-	-	-	-0.3563	1.9530
ϕ_7	-	-	-	-	-	-	-	-	-	-	-	
θ_{10}	-328.351	0.2302	7,618.1000	0.3182	0.6107	0.3457	0.3626	0.1929	-0.8152	0.1908	17.8136	29.8383
θ_{11}	-160.294	0.9647	-1,196.8000	0.5750	-0.5527	0.7197	-1.1847	0.1659	-1.2356	0.3808	3.5103	34.6796
θ_{12}	-336.763	0.6516	-17.7000	1.5695	3.5177	0.8990	2.2391	0.3769	2.8006	0.8309	-4.9874	3.6295
θ_{13}	107.133	1.5158	-57.3000	1.8193	-3.6285	1.1869	-1.2748	0.3821	-1.7257	0.7710	4.2078	3.8702
θ_{14}	-53.4889	2.4075	90.6000	2.1883	2.3600	1.0980	0.3345	0.3389	0.3700	0.6074	-2.3867	3.5481
θ_{15}	130.2654	1.5391	31.1000	2.2599	-1.5697	0.9668	0.2752	0.1738	0.2100	0.2815	1.4728	3.1431
θ_{16}	-	-	-	-	0.7868	0.7366	-	-	-	-	0.7243	2.1704
θ_{17}	-	-	-	-	-	-	-	-	-	-	-	
θ_{20}	-98.2139	2.6406	-147.2963	0.8512	-0.0045	0.0128	-	-	-	-	-	-
θ_{21}	26.5350	0.7790	96.2139	1.6993	0.3015	0.3592	-	-	-	-	-	-
θ_{22}	79.8829	1.1839	-6.6805	3.0229	-1.8244	0.4337	-	-	-	-	-	-
θ_{23}	-15.5313	1.2734	20.4282	3.1806	1.8168	0.5167	-	-	-	-	-	-
θ_{24}	6.9089	1.5679	-17.8232	2.7499	-1.1288	0.4262	-	-	-	-	-	-
θ_{25}	-37.3372	1.3244	3.8179	1.2654	0.6836	0.3218	-	-	-	-	-	-
θ_{26}	-	-	-	-	-0.2950	0.2336	-	-	-	-	-	-
θ_{27}	-	-	-	-	-	-	-	-	-	-	-	-
γ_1	0.6114	0.2302	3.0383	0.0331	1.0173	0.3457	0.0005	0.1229	0.0001	0.1908	0.0001	19.8383
γ_2	1.0247	0.8703	3.7500	0.0492	19.5033	10.9862	-0.0001	0.1259	-0.0001	0.3808	0.0008	24.6796
γ_3	0.2893	0.8703	3.0452	0.0333	-	-	-	-	-	-	-	-
c_1	2.1121	-	0.9085	0.0180	-0.0410	0.7197	2.0109	3769	2.7331	0.8309	-0.0761	3.6295
c_2	2.1673	-	0.9214	0.0146	0.0406	0.0450	-	-	-	-	-	-
c_3	2.8470	-	0.9267	0.0185	-	-	-	-	-	-	-	-

Diagnostics (p-values)

Diagnostic Test	STAR			GSTAR		
No error autocorrelation						
q=1	0.8947	0.7664	0.7563	0.0031	0.0001	0.3836
q=2	0.9913	0.9569	0.9530	0.0125	0.0004	0.6842
q=4	1.0000	0.9991	0.9989	0.0673	0.0034	0.9439
q=10	1.0000	1.0000	1.0000	0.5558	0.1089	1.0000
No rem. nonlinearity / No rem. asymmetry	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
Parameter constancy						
H1	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994
H2	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991
H3	1.0000	1.0000	1.0000	1.0000	1.0000	0.9991

NOTE: (a) Jarque-Brera ($p-$value) (b) Engle's test ($p-$value); (c) Durbin-Watson test ($p-$value); (d) Kolmogorov-Smirnov test ($p-$value); (1)Values expressed in test-statistics, significance denoted by '*' (5%), '**' (1%); (2) Results acquired by JMulTi; (3)Results acquired by RATS

Table 10: Predictive performances of GSTAR model for different forecast horizons and comparison with (MR)-STAR and AR models

Forecast horizon	Forecast Error Measure	AR IP	AR UN	AR YSSN	AR LYNX	(MR)STAR IP	(MR)STAR UN	(MR)STAR YSSN	(MR)STAR LYNX	GSTAR IP	GSTAR UN	GSTAR YSSN	GSTAR LYNX
Point predictive performances													
1	MFE	-0.0011	0.0009	-0.0300	0.0029	-0.0065	0.0156	0.0807	-0.0013	-0.0093	0.0095	0.0744	-0.0003
3		-0.1108	0.0340	-0.8837	-3.2369	-0.0071	0.0168	0.0975	-0.0058	-0.0098	0.0108	0.0881	-0.0031
6		-0.3024	0.0607	-0.0221	-8.6402	-0.0072	0.0150	0.1007	-0.0101	-0.0099	0.0093	0.0986	-0.0066
12		-1.4908	0.1117	-7.8745	-25.2732	-0.0068	0.0163	0.0984	-0.0176	-0.0094	0.0107	0.0957	-0.0115
1	sMAE	0.0011	0.0058	0.0009	0.0008	0.0011	0.0061	0.0008	0.0008	0.0011	0.0060	0.0008	0.0010
3		0.0037	0.0087	0.0016	0.0116	0.0011	0.0059	0.0008	0.0008	0.0011	0.0058	0.0008	0.0010
6		0.0070	0.0128	0.8572	0.0203	0.0012	0.0056	0.0008	0.0009	0.0012	0.0055	0.0008	0.0011
12		0.0124	0.0190	0.0050	0.0308	0.0012	0.0050	0.0009	0.0011	0.0012	0.0049	0.0009	0.0013
1	mRAE	1.0000	1.0000	1.0000	1.0000	1.0324	1.0224	1.7950	1.8697	1.0290	1.0107	1.6913	1.6567
3		1.0000	1.0000	1.0000	1.0000	1.1560	1.0552	2.0764	9.6346	1.1241	1.0369	2.1851	8.2831
6		1.0000	1.0000	1.0000	1.0000	1.4886	1.2189	2.5357	15.3676	1.4546	1.1693	2.7649	13.2310
12		1.0000	1.0000	1.0000	1.0000	2.0115	1.4965	9.4989	0.0260	2.1142	1.5085	9.4308	13.0395
1	RMSFE	0.0698	0.2856	1.0050	0.1878	0.0041	0.0178	0.0860	0.0243	0.0042	0.0176	0.0868	0.0283
3		0.2894	0.4234	2.0443	3.2842	0.0042	0.0179	0.0863	0.0249	0.0043	0.0177	0.0874	0.0291
6		0.7552	0.5915	1.8299	8.7639	0.0042	0.0181	0.0876	16.9274	0.0043	0.0179	0.0885	0.0305
12		0.7638	0.8545	9.0253	25.6618	0.0042	0.0184	0.0907	0.0281	0.0043	0.0182	0.0917	0.0336
Density predictive performances													
1	LogS	0.0020	0.0031	0.0073	0.0064	0.0017	0.0033	0.0072	0.0065	0.0018	0.0036	0.0072	0.0048
3		0.1108	0.0031	0.0072	0.0062	0.0017	0.0033	0.0072	0.0061	0.0018	0.0035	0.0072	0.0047
6		0.0020	0.0031	0.0071	0.0067	0.0017	0.0033	0.0072	0.0064	0.0018	0.0035	0.0071	0.0050
12		0.0020	0.0031	0.0071	0.0067	0.0017	0.0033	0.0072	0.0063	0.0018	0.0035	0.0072	0.0051
1	QS	1.1568	0.7986	0.2619	1.4829	2.1303	0.8511	0.2340	1.3515	2.1689	0.8683	0.2269	1.1717
3		1.1644	0.7998	0.2624	1.4698	2.1386	0.8537	0.2333	1.3384	2.1821	0.8698	0.2276	1.1385
6		1.1665	0.8125	0.2620	1.5058	2.1391	0.8550	0.2357	1.3762	2.1895	0.8698	0.2268	1.1677
12		1.2024	0.6741	0.2613	1.6211	2.1479	0.6412	0.2340	1.3106	2.1921	0.6354	0.2271	1.1668
1	CRPS	0.2142	1.6750	4.2628	1.5977	0.2170	1.6090	4.6027	1.4743	0.2201	1.6094	4.2893	1.4906
3		0.2161	1.6850	4.2932	1.6689	0.2189	1.6193	4.2935	1.5435	0.2224	1.6197	4.3243	1.5545
6		0.2188	1.6983	4.3551	1.7700	0.2212	1.6317	4.3787	1.6279	0.2252	1.6344	4.3942	1.6375
12		0.2255	1.7295	4.5782	2.0151	0.2279	1.6594	4.6027	1.8482	0.2318	1.6617	4.6064	1.8558
1	qS	0.0084	-0.0713	0.6198	0.2557	0.0089	-0.0727	0.6060	0.2571	0.0087	-0.0723	0.6067	0.2468
3		0.0083	-0.0706	0.6267	0.2527	0.0085	-0.0721	0.6119	0.2544	0.0086	-0.0717	0.6122	0.2439
6		0.0082	-0.0695	0.6329	0.2511	0.0083	-0.0708	0.6176	0.2537	0.0085	-0.0705	0.6173	0.2434
12		0.0077	-0.0667	0.6218	0.2455	0.0071	-0.0682	0.6068	0.2496	0.0080	-0.0679	0.6065	0.2387

Table 11: Predictive performances of GSTAR model under uncertainty for different forecast horizons and comparison with (MR)-STAR and AR models

		Point predictive performances											
Forecast horizon	Forecast Error Measure	AR				(MR)STAR				GSTAR			
		IP	UN	ENT	EXT	IP	UN	ENT	EXT	IP	UN	ENT	EXT
1		0.7134	-0.0309	0.0154	-0.0036	0.5894	-0.0308	0.0151	-0.0075	0.5781	0.0008	0.0225	0.0174
3	MFE	0.8400	-0.0006	0.0821	-0.0345	0.7628	-0.0205	0.0814	0.0641	0.5657	0.0465	0.0827	-0.0138
6		0.5048	-0.0020	0.1774	0.0068	0.7489	-0.0166	0.1481	-0.0098	0.6159	-0.0074	0.1586	-0.0689
12		0.8711	-0.0530	0.1779	-0.1237	0.8340	-0.0129	0.1998	-0.1437	0.7099	-0.0090	0.2649	-0.1629
1		0.0057	0.0308	0.1857	0.2001	0.0020	0.0144	0.1722	0.2085	0.0058	0.0272	0.1771	0.1753
3	sMAE	0.0056	0.0271	0.2892	0.1743	0.0020	0.0137	0.2552	0.1110	0.0056	0.0318	0.2873	0.1900
6		0.0056	0.0253	2.8832	0.1078	0.0021	0.0131	-2.0417	0.1172	0.0061	0.0258	-2.8480	0.1462
12		0.0059	0.0261	-0.5257	0.1049	0.0022	0.0124	-0.9993	0.1185	0.0056	0.0231	1.0605	0.1208
1		0.6853	0.0113	0.0069	0.0035	0.6814	0.0063	0.0057	0.0047	0.6143	0.0173	0.0080	0.0053
3	mRAE	0.5457	0.1735	0.0908	0.0679	1.1191	0.1223	0.1008	0.1624	0.9089	0.1175	0.0978	0.0675
6		0.9579	0.1774	0.1043	0.1510	3.0881	0.1245	0.0845	0.1171	0.9029	0.1701	0.0900	0.0744
12		1.7966	0.1514	0.0649	0.0849	5.1183	0.1421	0.0969	0.0682	0.9937	0.2119	0.1372	0.0615
1		0.0641	0.0780	0.1711	0.1965	0.0368	0.0603	0.1714	0.1960	0.0598	0.0774	0.1712	0.1964
3	RMSFE	0.0662	0.0789	0.1696	0.1999	0.0329	0.0604	0.1702	0.1991	0.0543	0.0781	0.1698	0.1982
6		0.0489	0.0784	0.1673	0.2040	0.0425	0.0606	0.1670	0.2047	0.0510	0.0786	0.1671	0.2044
12		0.0706	0.0797	0.1790	0.2103	0.0483	0.0612	0.1795	0.2103	0.0625	0.0800	0.1801	0.2110
		Density predictive performances											
1		0.0019	0.0036	0.0096	0.0148	-0.0001	0.0034	0.0096	0.0152	0.0020	0.0037	0.0098	0.0148
3	LogS	0.0018	0.0035	0.0097	0.0153	-0.0001	0.0034	0.0095	0.0156	0.0023	0.0039	0.0100	0.0150
6		0.0018	0.0035	0.0098	0.0156	-0.0001	-0.0166	0.0096	0.0159	0.0029	0.0040	0.0109	0.0156
12		0.0018	0.0037	0.0100	0.0158	-0.0001	0.0034	0.0099	0.0162	0.0038	0.0044	0.0110	0.0158
1		2.2142	0.8870	0.6235	0.2907	2.5261	1.0937	0.6232	0.2904	2.2040	0.8761	0.6235	0.2907
3	QS	2.2146	0.8879	0.6211	0.2870	2.4917	1.0995	0.6208	0.2868	2.2086	0.8769	0.6211	0.2888
6		2.2234	0.8876	0.6207	0.3060	2.1880	1.0929	0.6200	0.3057	2.2100	0.8773	0.6207	0.3060
12		2.2250	0.8920	0.6221	0.3207	2.2437	1.0095	0.6219	0.3205	2.2195	0.8819	0.6221	0.3207
1		-0.4926	1.6349	4.1190	4.8864	0.1968	0.8900	4.1648	4.6767	0.2219	1.6349	4.1190	4.8864
3	CRPS	0.2241	1.6431	4.2632	4.9607	0.1974	0.8922	4.3105	4.7493	0.2241	1.6431	4.2632	4.9229
6		0.2273	1.6579	4.1609	5.0927	0.1982	0.8954	4.2108	4.8670	0.2273	1.6579	4.1609	5.0927
12		0.2337	1.6865	4.5182	5.4112	0.1999	0.9018	4.5764	5.1850	0.2337	1.6865	4.5182	5.4112
1		0.0017	-0.0157	-0.0066	-0.0022	-0.0030	-0.0142	-0.0064	-0.0018	0.0019	-0.0145	-0.0066	-0.0012
3	qS	0.0017	-0.0155	-0.0079	-0.0014	-0.0030	-0.0141	-0.0077	-0.0011	0.0019	-0.0144	-0.0079	-0.0009
6		0.0019	-0.0153	-0.0091	-0.0005	-0.0030	-0.0140	-0.0089	-0.0002	0.0020	-0.0141	-0.0091	-0.0005
12		0.0021	-0.0146	-0.0103	0.0010	-0.0031	-0.0138	-0.0101	0.0015	0.0025	-0.0136	-0.0103	0.0014

Table 12: Comparison of GSTAR model's predictive ability with AR and MR-STAR model for three different tests.

	MR-STAR vs AR				GSTAR vs MR-STAR			
SERIES	IP	UN	ENT	EXT	IP	UN	ENT	EXT
h	Diebold-Mariano							
1	0.0000	0.0000	0.0000	0.2462	0.0000	0.0000	0.0000	0.2915
2	0.0003	0.0005	0.0208	0.3476	0.0000	0.0000	0.0266	0.2977
4	0.0012	0.0019	0.0045	0.3751	0.0000	0.0000	0.0056	0.3538
12	0.0026	0.0038	0.0000	0.3457	0.0000	0.0002	0.0000	0.2915
h	Giacomini-Whight							
1	0.0000	0.0000	0.0060	0.0042	0.0000	0.0000	0.0096	0.8142
2	0.0000	0.0000	0.1661	0.0954	0.0000	0.0000	0.1249	0.9116
4	0.0066	0.0072	0.1130	0.1016	0.0040	0.0215	0.3013	0.9230
12	0.0106	0.0133	0.1169	0.1820	0.0170	0.0420	0.2667	0.8142

Scoring Rule	Amisano-Giacomini															
h	IP		UN		ENT		EXT		IP		UN		ENT		EXT	
QSR	t	p-val	t	p-val	t	p-val	t	p-val	t	p-val	t	p-val	t	p-val	t	p-val
1	5.1057	<0.001	4.8320	<0.001	-0.2489	0.5981	-0.2223	0.5878	-5.1057	1.0000	4.2904	<0.001	0.2489	0.4018	0.2105	0.4121
2	0.3733	0.7067	4.9038	<0.001	-1.4816	0.9297	-0.4370	0.7806	0.3733	0.3545	4.5631	<0.001	1.4816	0.0702	0.2520	0.3560
4	-0.5440	0.2150	5.1023	0.0083	-1.3892	0.9165	-0.9422	0.8335	0.5440	0.2932	4.6200	0.0250	1.3892	0.0834	0.2597	0.3661
12	5.0349	<0.001	5.7302	0.0152	-1.4250	-0.9349	-1.3420	0.8397	-5.0349	1.0000	-1.0039	0.9231	-0.2458	0.5969	0.4812	0.7725
LogS																
1	650.28	1.0000	602.34	1.0000	0.2875	0.0000	2.2e04	<0.001	649.281	<0.001	43.764	0.6773	-2.8e04	1.0000	2.2e04	<0.001
2	-192.36	0.8899	-132.48	0.7530	0.2489	0.4018	2.1e04	<0.001	-192.364	1.0000	30.450	0.3502	-3.7e04	1.0000	2.2e04	<0.001
4	-3.7e03	1.0000	-2.2e03	1.0000	5.4e04	<0.001	3.4e04	<0.001	3.7e03	<0.001	12.304	0.1339	5.4e04	1.0000	2.5e04	<0.001
12	-641.28	1.0000	-459.23	1.0000	5.3e04	<0.001	3.2e04	<0.001	641.2803	<0.001	12.439	0.1010	3.5e04	1.0000	2.4e04	<0.001
CRPS																
1	-0.41249	0.6599	-0.2995	0.3028	-41.147	1.0000	-1.1293	0.8692	0.41249	0.3400	-0.3702	0.5265	41.146	<0.001	-1.0127	0.1302
2	-1.2273	0.1100	-0.4102	0.2504	42.153	1.0000	-2.0829	0.8706	-1.2273	0.8899	-0.7894	0.2028	42.153	<0.001	-1.2306	0.1665
4	2.2718	0.0116	-0.1501	0.2050	35.581	1.0000	-4.2301	0.9530	-2.2718	0.9883	-1.1490	0.0922	25.925	<0.001	-1.4875	0.2305
12	-0.4067	0.6578	-0.9501	0.5699	41.091	1.0000	-5.5210	0.9934	0.4067	0.3421	-1.4670	0.0579	0.3928	0.0340	-2.4014	0.3490
QuantS																
1	7.3000	1.0000	5.7031	0.9940	10.7e06	<0.001	-2.3e04	1.0000	-7.3029	1.0000	4.2499	0.3401	-5.3e03	1.0000	-2.3e03	1.0000
2	7.0429	1.0000	5.2301	0.9024	-10.7e06	1.0000	-2.3e04	1.0000	7.0429	1.0000	3.2993	0.2370	5.1e03	<0.001	-2.3e03	1.0000
4	8.7815	<0.001	2.1027	0.4501	7.1e06	1.0000	-24.445	1.0000	-8.7815	1.0000	2.3409	0.1502	7.5e03	<0.001	-2,5e03	1.0000
12	7.1988	<0.001	1.8235	0.3501	-1.1e06	1.0000	-24.780	1.0000	-7.1988	1.0000	2.9302	0.0829	-1.1e05	1.0000	-2,6e03	1.0000

NOTE: all the tests consider density forecasts generated from real data, according to the model estimated in Table 7 – 8. In the Amisano-Giacomini test, the GSTAR has the role of $\bar{S}^f$ and the benchmark (MR-ST)AR density forecast the role of $\bar{S}^g$. Since here LogS has positive orientation, if t-statistic is positive, f is preferred; the weight is assumed 1.