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Estimation of long memory in integrated variance

Eduardo Rossi
(Università di Pavia)

Paolo Santucci de Magistris
(Università di Padova e CREATES)

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Via San Felice, 5
I-27100 Pavia
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Estimation of long memory in integrated variance

Eduardo Rossi *
University of Pavia, Italy

Paolo Santucci de Magistris †
University of Padova and CREATES

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Abstract

A stylized fact is that realized variance has long memory. We show that, when the instantaneous volatility is a long memory process of order d, the integrated variance is characterized by the same long-range dependence. We prove that the spectral density of realized variance is given by the sum of the spectral density of the integrated variance plus that of a measurement error, due to the sparse sampling and market microstructure noise. Hence, the realized volatility has the same degree of long memory as the integrated variance. The additional term in the spectral density induces a finite-sample bias in the semiparametric estimates of the long memory. A Monte Carlo simulation provides evidence that the corrected local Whittle estimator of Hurvich et al. (2005) is much less biased than the standard local Whittle estimator and the empirical application shows that it is robust to the choice of the sampling frequency used to compute the realized variance. Finally, the empirical results suggest that the volatility series are more likely to be generated by a nonstationary fractional process.

Keywords: Realized variance, Long memory stochastic volatility, Measurement error, local Whittle estimator.
J.E.L: classification: C14, C22, C58.

*Dipartimento Economics and Management. Via San Felice 5, 27100 Pavia, Italy. Tel.: +39 0382/986207 Fax: +39 0382/304226. E-mail: `erossi@eco.unipv.it`.

†**Corresponding author:** School of Economics and Management, Aarhus University, Bartholins Alle 10, 1322 University Park, DK-8000 Aarhus C, Denmark; phone: +45 89 421 547; email address: `psantucci@creates.au.dk`

1 Introduction

A well documented stylized fact is that the volatility of the financial returns is characterized by long-range dependence, or long memory, see, for instance, Baillie (1996), Bollerslev and Mikkelsen (1996), Dacorogna et al. (1993), Ding et al. (1993), Granger and Ding (1996). More recently Andersen et al. (2001a), Andersen et al. (2001b), Andersen et al. (2003), Martens et al. (2009) report evidence of stationary long memory in the realized variance (or realized volatility, RV) series.

In this paper, we theoretically study the long memory properties of the integrated variance (IV) and RV, assuming that the instantaneous volatility, $\sigma^2(t)$, is characterized by long memory of order d.

The contributions of this paper are threefold. Firstly, we demonstrate that IV has the same fractional integration order of $\sigma^2(t)$, since it has a pole at the zero frequencies that depends on the long memory parameter, d. This result holds for both stationary and nonstationary long memory stochastic volatility models. Secondly, we show that when we consider sparse sampling and the presence of market microstructure noise, see Bandi and Russell (2008), Hansen and Lunde (2006) and for a recent survey McAleer and Medeiros (2008), the spectral density of RV is given by the spectral density of IV plus an additional constant term, which depends on the variance of the measurement error term. Therefore, RV is also a long-range dependent process and it has the same long memory of IV and $\sigma^2(t)$. Moreover, in absence of microstructure noise, the spectral density of RV converges to that of IV, as the sampling frequency increases.

Thirdly, we show by simulation that the local Whittle (LW) estimator of the long memory parameter is biased in finite samples as a consequence of the presence of the measurement error in the spectral density of RV. In the context of our signal plus noise model, an alternative choice to the LW estimator is the corrected LW estimator of Hurvich et al. (2005), that explicitly accounts for the presence of the measurement error. We evaluate the impact that the choice of the sampling frequency and the variance of the measurement error have on the finite sample bias and the variability of both estimators. The results highlight the dramatic reduction of the finite-sample bias of the corrected LW estimator with respect to the standard LW estimator.

The effectiveness of corrected LW estimator is also evident from the estimates of the degree of long memory of the volatility of 28 stocks traded on the NYSE. The results obtained with RV at different sampling frequencies confirm the robustness of the corrected LW estimator to the presence of the measurement error, unlike the LW estimator, which is affected by the choice of the sampling frequency since this impacts on the variance of the measurement error. Indeed, the LW estimates decrease as the sampling frequency is getting smaller. The corrected LW estimates are not only larger than the uncorrected ones, but also rather constant with respect to the sampling frequency. This is particularly evident from the *long memory signature plot* of the RV, namely the plot of the long memory estimates obtained with RV at different sampling frequencies. The corrected estimates are always larger than 1/2, which is the upper bound of the stationary region. These results suggest that a deeper analysis of the nonstationary volatility models is called for.

This paper is organized as follows. In Section 2 we show that, when the instantaneous volatility is driven by a fractional Brownian motion, the degree of fractional integration of the IV process is the same as the instantaneous volatility. Section 2.1 illustrates the characteristics of the measurement error. Section 3 discusses the semiparametric techinque to obtain unbiased estimates of the long memory parameter of IV, based on a careful characterization of the spectral density of the realized variance. In Section 3.1 the results of the Monte Carlo simulations are illustrated and discussed. Section 4 compares the results obtained with the corrected and the uncorrected estimators of long memory, based on RV estimated on the real data. Section 5 concludes.

2 Long memory in integrated and realized variance: theoretical results

Let $P(t)$ be the price of an asset, where its logarithm, $p(t)$, follows the stochastic differential equation:

$$\mathrm{d}p(t) = m(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t) \tag{1}$$

where $W(t)$ is a standard Brownian motion and $m(t)$ is locally bounded and predictable. $\sigma^2(t)$ is assumed to be independent of $W(t)$ and càdlàg, see Barndorff-Nielsen and Shephard (2002a,b). Moreover, it is assumed that $\sigma^2(t)$ is a long memory process, such that

$$f_{\sigma^2}(\lambda) \sim c\lambda^{-2d} \quad \text{as} \quad \lambda \to 0 \tag{2}$$

where $c \in \mathbb{R}_+$ and $f_\sigma(\lambda)$ is the spectral density of $\sigma^2(t)$. When $0 < d < 1/2$, the process $\sigma^2(t)$ is stationary, while when $1/2 \leq d < 1$, the process $\sigma^2(t)$ is nonstationary, see Solo (1992), Velasco (1999a,b), Hurvich and Ray (1995, 2003) and Hurvich et al. (2005). In the nonstationary long memory case, the spectral density is not defined and it is replaced by the so called *pseudo* spectral density, which is the limit of the expectation of the sample periodogram. As noted by Hurvich and Ray (2003), the pseudo-spectral density plays a similar role as the ordinary spectral density in determining the properties of the periodogram, when $d > 1/2$, see also Hurvich et al. (2005, p.1288).

An example of a stationary long memory process for $\sigma^2(t)$ that satisfies condition (2) is the fractional Ornstein-Uhlenbeck process of Comte and Renault (1998):

$$\mathrm{d}\ln\sigma^2(t) = -k\ln\sigma^2(t)\mathrm{d}t + \gamma\mathrm{d}W_d(t) \tag{3}$$

where $k > 0$ is the drift parameter, while $\gamma > 0$ is the volatility parameter and

$W_d(t)$ is the fractional Brownian motion (fBm), which is defined[1] as

$$W_d(t) = \frac{1}{\Gamma(1+d)} \int_0^t (t-s)^d \mathrm{d}W(s) + \int_{-\infty}^0 [(t-s)^d - (-s)^d] \mathrm{d}W(s) \qquad (4)$$

The solution of (3) can be written as $\ln \sigma^2(t) = \int_0^t e^{-k(t-s)} \gamma \, \mathrm{d}W_d(s)$. In particular, for $0 < d < 1/2$, Comte (1996) and Comte and Renault (1998) show that the process $\ln \sigma^2(t)$ has long memory of order d, if $k_\infty = \gamma d/k$ is finite and different from zero. Comte and Renault (1998) show that, when $k > 0$, the volatility process $\sigma^2(t)$ is asymptotically equivalent (in quadratic mean) to the stationary process[2]:

$$\tilde{\sigma}^2(t) = \exp\left(\int_{-\infty}^t e^{-k(t-u)} \gamma \, \mathrm{d}W_d(u) \right), \quad k > 0 \quad 0 < d < \frac{1}{2}. \qquad (5)$$

They prove that the spectral density, $f_{\tilde{\sigma}^2}(\lambda)$, of the process $\tilde{\sigma}^2(t)$, is proportional to λ^{-2d} as $\lambda \to 0$, so that the volatility process inherits the long-memory property induced by the fBm. More recently, Comte et al. (2010) show that also the class of affine fractional stochastic volatility models, satisfies condition (2).

In the following proposition, we show that the IV has the same long memory degree of the instantaneous volatility.

Proposition 1 *Assuming that $\sigma^2(t)$ has a spectral density (or pseudo spectral density) that admits the representation in* (2) *around 0, then* $\lim_{\lambda \to 0} \lambda^{2d} f_{IV}(\lambda) = c \in \mathbb{R}_+$ *where $f_{IV}(\lambda)$ is the spectral density (or pseudo spectral density) of* $IV_t = \int_{t-1}^t \sigma^2(u) \mathrm{d}u$.

It is interesting to note that this result is proved under the assumption that $\sigma^2(t)$ has a spectral density (or pesudo spectral density) proportional to λ^{-2d} at the origin as in (2). As a consequence of Proposition (1), at the origin the spectral density (or pseudo spectral density) of the IV has the same behavior as that of the instantaneous volatility, so that IV has the same degree of long memory as $\sigma^2(t)$. This result holds for stationary and nonstationary long memory instan-

[1] The literature on long memory processes in econometrics distinguishes between type I and type II fractional Brownian motion. These processes have been carefully examined and contrasted by Marinucci and Robinson (1999) and Davidson and Hashimzade (2009).

[2] The volatility process $\sigma^2(t)$ coincides almost surely with $\tilde{\sigma}^2(t)$.

taneous volatilities. Strictly speaking, finding a degree of long memory of IV larger than 1/2 implies an instantaneous volatility of the same fractional order. Although a model specification for the instantaneous volatility that allows for nonstationary long memory is potentially interesting and empirically relevant, nonstationary long memory continuous time stochastic volatility models are not yet investigated in literature.[3] In Section 3.1, we propose a nonstationary long memory process for the instantaneous volatility, and we study by simulations the consequences of nonstationarity on the long memory of IV. In the next section, we analyze how the presence of long memory in the IV translates in long-range dependence of the RV.

2.1 The measurement error

In this section we characterize the measurement error associated with the RV estimator. To simplify the notation, we consider an equidistant partition $0 = t_0 < t_1 < \ldots < t_n = 1$, where $t_i = i/n$, and $\Delta = 1/n$, that is the interval is normalized to have unit length. Define $r_{i,\Delta} = p_{i\Delta,\Delta} - p_{(i-1)\Delta,\Delta}$. Adopting the notation of Hansen and Lunde (2005), the RV^{Δ} at sampling frequency Δ is

$$RV^{\Delta} = \sum_{i=1}^{n} r_{i,\Delta}^2 \tag{6}$$

and Barndorff-Nielsen and Shephard (2002b) derived a distribution theory for RV^{Δ} when $n \to \infty$,

$$\sqrt{n}(RV^{\Delta} - IV) \xrightarrow{d} N(0, 2IQ),$$

where $IQ = \int_0^1 \sigma^4(u)\mathrm{d}u$ is the *integrated quarticity*. In this paper we focus on the series of non-overlapping IV, $\{IV_t\}_{t=1}^{T}$, where $[0,T]$ represents our sampling period. Further, the time series of non-overlapping RV^{Δ} is composed by $\left\{RV_t^{\Delta} = \sum_{i=1}^{n} r_{t,i,\Delta}^2\right\}_{t=1}^{T}$, where $r_{t,i,\Delta} = p_{t-1+i\Delta,\Delta} - p_{t-1+(i-1)\Delta,\Delta}$.[4] We are interested in the estimation of the long memory of IV_t, using the observations on

[3]For example, the estimates of long memory presented in Comte and Renault (1998) are larger than 1/2.

[4]In order to obtain non-overlapping RV_t^{Δ}'s the first return included in the computation of RV_t^{Δ} is $r_{t,2,\Delta} = p_{t-1+2\Delta,\Delta} - p_{t-1+\Delta,\Delta}$.

RV_t^{Δ}.

2.1.1 No Noise Case

Barndorff-Nielsen and Shephard (2002b) and Meddahi (2002) characterize the discretization error, when RV is used to measure the IV. While RV^{Δ} converges to IV when $\Delta \to 0$, the difference may be not negligible for a given $\Delta > 0$. Following Barndorff-Nielsen and Shephard (2002b) and Meddahi (2002), we can decompose RV_t^{Δ}, for a given Δ, as

$$RV_t^{\Delta} \quad = \quad IV_t + u_t^{\Delta}. \tag{7}$$

with the *discretization error* equal to

$$u_t^{\Delta} = \sum_{i=1}^{n} u_{t,i}^{\Delta}, \tag{8}$$

where $u_{t,i}^{\Delta} = r_{t,i,\Delta}^2 - \sigma_{t,i,\Delta}^2$ is the discretization error in th i-th subinterval, with $\sigma_{t,i,\Delta}^2 = \int_{t-1+(i-1)\Delta}^{t-1+i\Delta} \sigma^2(u)\mathrm{d}u$. When the drift $m(t)$ is non-zero, Meddahi (2002) proves that u_t^{Δ} has a non-zero mean. Furthermore, as pointed out by Barndorff-Nielsen and Shephard (2002b) and Meddahi (2002), the correlation between the IV_t and the noise term is zero when there is no leverage effect, that is $\mathrm{d}W$ in (1) and $\mathrm{d}W_d$ in (3) are uncorrelated processes.

In the next Proposition we characterize the properties of the spectral density of RV^{Δ}.

Proposition 2 *Consider the processes for $p(t)$ and RV_t^{Δ} defined in* (1) *and* (6)*. Assume that condition* (2) *holds for $\sigma^2(t)$. Let $m(t) = \mu$ and assume no leverage effect, $\rho = corr(dW(t), dW_d(t)) = 0$,*

i. For $\Delta > 0$, the spectral density (or pseudo spectral density) of RV_t^{Δ} is given by

$$f_{RV^{\Delta}}(\lambda) = f_{IV}(\lambda) + f_{u^{\Delta}}(\lambda) \tag{9}$$

and $\lim_{\lambda\to 0} \lambda^{2d} f_{RV^{\Delta}}(\lambda) = \lim_{\lambda\to 0} \lambda^{2d}(f_{IV}(\lambda) + f_{u^{\Delta}}(\lambda)) = c.$

ii. When $\Delta \to 0$, $\mathrm{Var}(u_t^{\Delta}) \to 0$, *then*

$$\lim_{\Delta \to 0} f_{RV^{\Delta}}(\lambda) = f_{IV}(\lambda). \tag{10}$$

And

$$\lim_{\lambda \to 0} \left[\lim_{\Delta \to 0} \lambda^{2d} f_{RV^{\Delta}}(\lambda) \right] = \lim_{\lambda \to 0} \lambda^{2d} f_{IV}(\lambda) = c. \tag{11}$$

For a given $\Delta > 0$, the spectral density (or pseudo spectral density) of RV_t^{Δ} is equal to that of IV_t plus an additional term which depends on the variance of u_t^{Δ}. This means that RV_t^{Δ} has the same degree of long memory of IV_t since $f_{u^{\Delta}}(\lambda) = \frac{\mathrm{Var}(u_t^{\Delta})}{2\pi}$ is constant with respect to λ. In the ideal situation where prices are recorded continuously ($\Delta \to 0$), the spectral density of RV_t^{Δ} converges to that of the IV_t and, again, they share the same degree of long memory. It should be noted that the results in Proposition 2 extend to any stochastic volatility process for which the spectral density (or the pseudo spectral density) of IV_t exists. Note that the results in (9) and (11) are valid when $m(t) = \mu \neq 0$ in (1), since the discretization error remains an uncorrelated process. However, in this case, a closed from expression for $\mathrm{Var}[u_t^{\Delta}]$ becomes more involved than that provided by Barndorff-Nielsen and Shephard (2002a), because it depends also on $m(t)$.[5]

For example, when the instantaneous volatility follows the process in (3) and $m(t) = 0$, as in Barndorff-Nielsen and Shephard (2002a, p.257), the $\mathrm{Var}[u_t^{\Delta}]$ term is equal to

$$\mathrm{Var}[u_t^{\Delta}] = 2\Delta^{-1} \cdot \left\{ 2\,\mathrm{Var}[\tilde{\sigma}^2(t)] \cdot \int_0^{\Delta} \int_0^{v} r(u)\mathrm{d}u\mathrm{d}v + \Delta^2 E[\tilde{\sigma}^2(t)]^2 \right\}, \tag{12}$$

where

$$E[\tilde{\sigma}^2(t)] = \exp\left(\frac{\omega^2}{2}\right), \quad \mathrm{Var}[\tilde{\sigma}^2(t)] = [\exp(\omega^2) - 1]\exp(\omega^2) \tag{13}$$

with $\omega^2 \equiv \mathrm{Var}[\ln \sigma^2(t)] = \frac{\gamma^2 \pi}{\Gamma^2(1+d)k^{1+2d}\cos(d\pi)}$, see Casas and Gao (2008), and

[5] Meddahi (2002) derives a closed form expression for $\mathrm{Var}[u_t^{\Delta}]$ for the class of eigenfunction stochastic volatility models and assuming that $m(t) = \mu$.

$r(\cdot)$ denotes the autocorrelation function of the process $\tilde{\sigma}^2(t)$. It is clear that the parameters in (3) affect $E[\tilde{\sigma}^2(t)]$, $\mathrm{Var}[\tilde{\sigma}^2(t)]$, and $\mathrm{Cov}(\tilde{\sigma}^2(t+h), \tilde{\sigma}^2(t))$, and, through these, impact on the variance of the discretization error. It is hard to obtain closed-form expressions for the partial derivatives of $\mathrm{Var}[u_t^\Delta]$ w.r.t. the parameters in (3), thus we investigate this point by simulations in Section 3.1.[6]

Differently from discrete-time stochastic volatility framework, where the variance of the measurement error is unrelated to the volatility parameters, in this setup the variance of the discretization error is a highly non-linear function of the instantaneous volatility process parameters. In the next section, we will characterize the measurement error and the spectral density of RV_t^Δ when the prices are contaminated by the microstructure noise.

2.1.2 Microstructure Noise Case

Suppose now that the intradaily price is observed with error, due to the presence of microstructure noise,

$$\widetilde{p}(t) = p(t) + \epsilon(t) \tag{14}$$

where $p(t)$ is the latent true, or efficient, price process that follows (1). The term $\epsilon(t)$ is the noise around the true price, with mean 0 and finite fourth moment. In particular, $\epsilon(t)$ is *i.i.d.* and it is independent of the efficient price and the true return process. Over periods of length Δ, we have

$$\tilde{r}_{t,i,\Delta} = \left(p_{t-1+i\Delta,\Delta} - p_{t-1+(i-1)\Delta,\Delta}\right) + \left(\epsilon_{t,i,\Delta} - \epsilon_{t,i-1,\Delta}\right) = r_{t,i,\Delta} + \eta_{t,i,\Delta}. \tag{15}$$

With discretization and microstructure noise, and $m(t) = \mu$, the measurement error of RV_t^Δ is given by

$$\xi_t^\Delta = u_t^\Delta + \sum_{i=1}^{n} \eta_{t,i,\Delta}^2 + 2\left(\sum_{j=1}^{n} \sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right) + 2\Delta\mu \sum_{j=1}^{n} \eta_{t,i,\Delta}. \tag{16}$$

As noted by Bandi and Russell (2006), while the efficient return is of order $O_p(\sqrt{\Delta})$, the microstructure noise is of order $O_p(1)$ over any period of time.

[6] The double integral in (12) can only be approximated for $\Delta \to 0$.

This means, that, when $\Delta \to 0$, the microstructure noise dominates over the true return process, and longer period returns are less contaminated by the noise than shorter period returns. Given the properties of $\epsilon(t)$, then

Proposition 3 *Consider the processes for $p(t)$, $\widetilde{p}(t)$ and RV_t^{Δ} defined respectively in* (1)*,* (14) *and* (6)*. Let $m(t) = \mu$ and assume no leverage effect, $\rho = corr(dW(t), dW_d(t)) = 0$,*

i. For $\Delta > 0$,

$$f_{RV^{\Delta}}(\lambda) = f_{IV}(\lambda) + f_{\xi^{\Delta}}(\lambda) \tag{17}$$

where $f_{\xi^{\Delta}}(\lambda) = \frac{\mathrm{Var}(\xi_t^{\Delta})}{2\pi}$ is the spectral density of the measurement error term. It follows that

$$\lim_{\lambda \to 0} \lambda^{2d} f_{RV^{\Delta}}(\lambda) = c, \tag{18}$$

with $c > 0$.

ii. For $\Delta \to 0$, $\mathrm{Var}(\xi_t^{\Delta}) \to \infty$ and the $f_{RV^{\Delta}}(\lambda) \to \infty$, $\forall \lambda$.

Proposition 3 extends the results of Proposition 2, namely RV_t^{Δ} is characterized by the same degree of long memory of the IV_t even when the microstructure noise is present and $\Delta > 0$. When prices are observed with a microstructure noise, the long memory signal, IV_t, turns out to be contaminated by a measurement error, whose variance is given by the sum of two components: the variance of the discretization error $\mathrm{Var}(u_t^{\Delta})$, and the term due to the presence of the microstructure noise, which is given by $\Delta^{-1}(E(\eta_{t,i,\Delta}^4) - \sigma_\eta^4) + 4\sigma_\eta^2 \Delta^{-1} E\left[\sigma_{t,i,\Delta}^2\right] + 4\Delta\mu^2\sigma_\eta^2$, see (37) in Appendix A.3. In accordance with Proposition 3, the effect of the microstructure noise on the variance of ξ_t^{Δ}, diverges as $\Delta \to 0$, see Bandi and Russell (2006), and dominates the long memory signal which cannot be identified anymore. However, for a given $\Delta > 0$, the $\mathrm{Var}(\xi_t^{\Delta})$ is finite, and RV_t^{Δ} has the same long memory degree of IV_t. On the other hand, the choice of Δ impacts on the variance of ξ_t^{Δ} and through this on the spectral density of RV^{Δ}. If we decrease Δ, this reduces the variability of ξ_t^{Δ} due to the discretization but increases the microstructure noise component, so that the net effect on $\mathrm{Var}(\xi_t^{\Delta})$ is unknown a priori. This trade-off, which depends on the choice of Δ, will be studied via simulations in Section 3.1.

3 Bias corrected estimation of long memory in integrated variance

Now, we turn our attention to the estimation of the long memory parameter d by means of periodogram-based estimators using a LW criterion function. It is well known that the drawback of global long memory estimators is that they require unnecessary assumptions on the spectral density. Instead, a consistent estimate of d can be obtained simply by specifying the shape of the spectral density at the origin. These methods are referred as local methods. Further, the semiparametric approach has the advantage, over the parametric ones, that it does not require a full specification of the dynamics of the process, but simply characterization of the spectrum as $\lambda \to 0$. This implies that semiparametric estimation is more robust to the misspecification of the dynamics.

In our case, we are interested in the estimation of the long memory of IV_t (or $\sigma^2(t)$), using the observations on RV_t^Δ. According to Proposition 3, the spectrum of RV_t^Δ for $\Delta > 0$ is that of IV_t, plus an additional term, which depends on the variance of the measurement error, see (37). This is a typical signal-plus-noise problem. Consequently, the quality of the semiparametric estimate of d, based on spectrum of RV_t^Δ, can be dramatically affected in finite samples by the variability of the measurement error. In a recent paper, Hansen and Lunde (2010) note that, *"'even with the most accurate estimators of daily volatility, which can utilize thousands of high-frequency prices, the standard error for a single estimate is rarely less than 10 %."'*

In this section, we discuss the effect of the variance of the measurement error on the semiparametric estimates of d. A large literature, see among others Deo and Hurvich (2001), Hurvich et al. (2005) and Haldrup and Nielsen (2007), discusses the properties of the semiparametric long memory estimators, such as the log-periodogram regression and the LW estimator, when the long memory signal is contaminated by a noise term.[7] Deo and Hurvich (2001) show that the Geweke and Porter-Hudak (1984) estimator is biased by a constant factor that depends on the variance of the noise term. Sun and Phillips (2003) introduce an additional nonlinear term in the log-periodogram regression, proportional to λ^{2d}, to account for the effect of the additive noise term, that is allowed to be weakly dependent. Arteche (2004) suggests that an optimal choice of the bandwidth is important to minimize the influence of the added noise term, since the variance of the measurement error heavily restricts the allowable bandwidth in finite samples. With a larger variance of the noise with respect to the signal, only the frequencies very close to the origin contain a valuable information. Arteche (2004) and Hurvich et al. (2005) show that, in the signal-plus-noise framework, the LW estimator is consistent for $d \in (0,1)$ under general assumptions on the noise term. However, in finite samples, the estimates are downward biased.

A possible solution to this problem is provided by Hurvich et al. (2005). They consider a semiparametric specification of the spectral density, allowing also for possible correlation between the noise and the signal. In particular, it is required that the signal has an infinite moving average representation with mild conditions on the coefficients. For example, when $\sigma^2(t)$ is generated by the process in (3), a closed form expression of the spectral density of the signal, the IV_t, is hard to obtain. However, in this case IV_t is a long memory stationary process which can be approximated by an infinite moving average representation.[8] The modified LW objective function of Hurvich et al. (2005) is

$$Q(G,d,\beta)^* = \frac{1}{m}\sum_{j=1}^{m}\left\{\ln\left[G\lambda_j^{-2d}(1+\beta\lambda_j^{2d})\right] + \frac{\lambda_j^{2d} I_{RV^\Delta}(\lambda_j)}{G(1+\beta\lambda_j^{2d})}\right\}, \qquad (19)$$

[7]In this Section, we will maintain the assumption that the noise term is dynamically uncorrelated with the signal and it is a white noise. As shown in section 2.1, this is the relevant when drift in price and leverage are excluded.

[8]Comte et al. (2010) analyze the affine fractional stochastic volatility models and characterize the autocovariance function of the expected IV_t process.

where G is the spectrum at the origin.[9] Concentrating G out, it yields

$$R(d,\beta) = \frac{1}{m}\sum_{j=1}^{m} \ln\left(\lambda_j^{-2d}(1+\beta\lambda_j^{2d})\right) + \ln\left(\frac{1}{m}\sum_{j=1}^{m}\frac{\lambda_j^{2d} I_{RV^\Delta}(\lambda_j)}{(1+\beta\lambda_j^{2d})}\right), \tag{20}$$

where the LW estimator is obtained setting $\beta = 0$ in the minimization of R. The LW estimates of d and β are

$$(\hat{d}_c, \hat{\beta}) = \arg \min_{(d,\beta)\in\mathbb{D}\times\mathbb{B}} \hat{R}(d,\beta) \tag{21}$$

where $\mathbb{D}$ and $\mathbb{B}$ are the admissible sets of d and β, and m has to tend faster to ∞ than $T^{4d/(4d+1)}$. In the case of RV, $\hat{\beta}$ is interpreted as an estimate of the noise-to-signal ratio, $\frac{\mathrm{Var}(\xi_t^\Delta)}{2\pi f_{IV}(0)}$. The corrected estimator is consistent for $d \in (0,1)$ and asymptotically normal for $d \in (0, 3/4)$ with asymptotic variance, in absence of correlation between the signal and the noise, equal to $\frac{(1+d)^2}{16d^2\cdot m}$, which is a decreasing function of d. It is interesting to note that, for all the admissible values of d, the asymptotic variance of the bias-corrected LW estimator, $\hat{d}_c$, is larger than the corresponding asymptotic variance of the LW estimator, $\hat{d}$, that is $\frac{1}{4\cdot m}$.

3.1 Simulations

In this section we present the results of the Monte Carlo analysis of the finite sample properties of the long memory estimation of IV_t based on RV_t^Δ. We want to investigate the impact that the measurement error has on the LW and corrected LW estimators of the long memory of IV_t, disentangling the contribution of the discretization error from that due to the microstructure noise. The purpose is to evaluate if the corrected LW estimator provides superior performances, in terms of bias for a large range of values of the noise-to-signal ratio (nsr), and for different choices of Δ.

[9] Frederiksen et al. (2012) and Nielsen (2008) suggest to approximate the log-spectrum of the short-memory component of the signal and of the perturbation by means of an even polynomial term.

3.1.1 Setup

We consider two alternative setups for the generation of $\sigma^2(t)$. The first is the stationary one presented in equation (3), which has been discussed so far. However, given that there is some empirical evidence that the volatility may be nonstationary, see for example Comte and Renault (1998) and Harvey (1998), we also simulate $\sigma^2(t)$ from a nonstationary long memory process.

We assume that the log-price $p(t)$ follows:

$$\mathrm{d}p(t) = \sigma(t)\mathrm{d}W(t) \tag{22}$$

and the instantaneous volatility process $\sigma^2(t)$ is either:

$$\mathrm{d}\ln\sigma^2(t) = k(\psi - \ln\sigma^2(t))\mathrm{d}t + \gamma\mathrm{d}W_d(t) \tag{23}$$

or

$$\mathrm{d}\ln\sigma^2(t) = \gamma\mathrm{d}W_d(t) \tag{24}$$

where W_d is the fractional Brownian motion of order d independent of $W(t)$. To simulate increments from the fractional Brownian motion we implement the Matlab routine by Yingchun Zhou and Stilian Stoev[10] which is based on the circulant embedding algorithm for the values of interest of the Hurst's exponent, $H = d + \frac{1}{2}$.

The specification in (23) has a stationary solution, such that the long memory is equal to d, while the one in (24) generates nonstationary volatility trajectories.

In fact, when $\gamma > 0$, in the neighborhood of $\gamma = 0$, the first order approximation of $\exp\{\gamma W_d(t)\}$ is $1 + \gamma W_d(t)$, so that, following Solo (1992), the pseudo spectral density of $\sigma^2(t)$ has a pole in zero which is proportional to $\lambda^{-2\delta}$, where $\delta = 1 + d$. Therefore, when $d \in (-1/2, 0)$, then $\delta \in (\frac{1}{2}, 1)$, so that $\sigma^2(t)$ is a nonstationary (but *"'mean reverting"'*) long memory process. According to Proposition 1, the IV_t also has a pseudo spectral density proportional to $\lambda^{-2\delta}$ at the origin and it is integrated of order $\delta > \frac{1}{2}$. We will evaluate in simulation

[10]http://www.stat.lsa.umich.edu/ sstoev/code/ffgn.m

if IV_t has the expected long memory degree.

We simulate from the Euler approximation of (22), (23) and (24), a set of discrete trajectories with a time step of 10 seconds for 6.5 hours per day, which roughly corresponds to the trading period of NYSE. Thus we have a total $6 \times 60 \times 6.5 = 2,160$ log-prices and log-instantaneous volatilities per day for $2,500$ days, that is $Y_j = \{p_j, \ln \sigma_j^2\}_{j=1}^{2,160\times 2,500}$. The generated price series are used to compute the RV series, with different Δ. Note that the computational burden is due to the fact that we simulate for each Monte Carlo replication a trajectory of Y_j which has $5,400,000$ observations. Therefore, we treat the instantaneous volatilities generated at 10 seconds frequency as the true latent process. We estimate the long memory parameter of the IV_t, which is unobserved in practice, but known in a simulation study and computed for the day t as $IV_t = \sum_{k=1}^{2,160} \sigma^2_{(t-1)\cdot 2,160+k}$ for $t = 1, \ldots, 2.500$. The estimate of the long memory parameter of IV_t is a natural benchmark for those based on RV_t^{Δ}. Moreover, when sampling prices at 10 seconds, and constructing the RV measure, RV^{all}, is equivalent to letting $\Delta \to 0$. On the other hand, sampling at 1, 5, 10 and 30 minutes introduces the discretization error, mentioned in Section 2, which is the consequence of the sparse sampling. Finally when prices are recorded with noise and the sampling is at 1, 5, 10 and 30 minutes, then we have the joint effect of microstructure noise and sparse sampling.

The market microstructure noise is introduced in the simulations in the form of a bid-ask bounce, modeled as:

$$\widetilde{p}(t) = p(t) + \frac{\zeta}{2}\mathbb{I}(t) \tag{25}$$

where ζ is the percentage spread, and the order-driven indicator variables $\mathbb{I}(t)$ are independently across p and t and identically distributed with $Pr\{\mathbb{I}(t) = 1\} = Pr\{\mathbb{I}(t) = -1\} = \frac{1}{2}$. This variable takes value 1 when the transaction is buyer-initiated, and -1 when it is seller-initiated. We adopt the simplest bid-ask bounce specification in order to make a comparison with the existing literature. Furthermore it is interesting to note that $\mathrm{d}\widetilde{p}(t)$ exhibits spurious volatility and negative serial correlation, see for instance Nielsen and Frederiksen (2008). The parameter of the bid-ask spread, ζ, is set according to the values found in Table 1 in Bandi and Russell (2006). We choose $\zeta = \{0.000, 0.001, 0.002\}$ which are common values to the most liquid stocks. Similar values for ζ are also used in Nielsen and Frederiksen (2008).

The two estimators of the long memory, the LW ($\hat{d}$) and the corrected LW ($\hat{d}_c$), are computed using the RV_t^Δ series obtained at different sampling frequencies, i.e. Δ = 10 sec, 1 min, 5 min, 10 min and 30 min. In order to compare their finite sample performances, we compute, for each sampling frequency, the percentage relative bias from S Monte Carlo simulations

$$\text{Bias}(\hat{d}) = \frac{100}{d}\left(\frac{1}{S}\sum_{s=1}^{S}(\hat{d}_s - d)\right), \tag{26}$$

and the RMSE

$$\text{RMSE}(\hat{d}) = \left(\frac{1}{S}\sum_{s=1}^{S}(\hat{d}_s - d)^2\right)^{1/2}. \tag{27}$$

3.1.2 Noise-to-signal ratio

A crucial quantity in the simulations is the *nsr*, $\frac{\text{Var}(\xi_t^\Delta)}{\text{Var}(IV_t)}$, which depends on the generating process parameters, as discussed in Section 2.1. To figure out this relationship, which obviously affects the simulation results, we use Monte Carlo simulations, and plot the simulated *nsr* as a function of d, γ, ζ, and Δ. In this way, we can choose a combination of the structural parameters that resembles realistic values of the *nsr*, see for a discussion Meddahi (2002).

Figure 1(a) shows the simulated *nsr* for different choices of d and Δ. It is evident that increasing Δ increases, for each choice of d, the *nsr*, provided that the microstructure noise is absent. For moderate choices of Δ, 1 or 5 minutes, the impact of d on the *nsr* is rather limited.

When the *nsr* is plotted for different γ's, Figure 1(b), it is clear that, for a given Δ, as γ approaches 0, the innovation in the price process becomes the prevailing source of variability, so that the *nsr* is shifted upwards. This is seen simply noting that,

$$\frac{\mathrm{Var}(\xi_{t,i}^{\Delta})}{\mathrm{Var}(\sigma_{t,i,\Delta}^{2})} = \frac{\mathrm{Var}(\xi_{t,i}^{\Delta})}{E[(\sigma_{t,i,\Delta}^{2})^{2}] - E[(\sigma_{t,i,\Delta}^{2})]^{2}}.$$

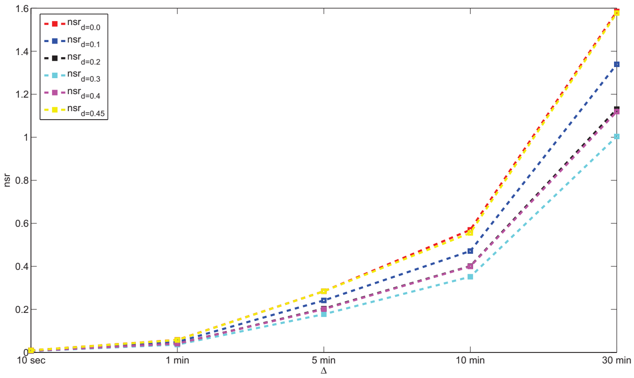
As $\gamma \to 0$, then $E[(\sigma_{t,i,\Delta}^{2})^{2}] - E[(\sigma_{t,i,\Delta}^{2})]^{2} \to 0$, so that $\frac{\mathrm{Var}(\xi_{t}^{\Delta})}{\mathrm{Var}(IV_t)} \to \infty$, where $\xi_{t,i}^{\Delta}$ is defined in (35). The *nsr* is increasing in Δ, starting from 1 minute frequency, while for 10 seconds the microstructure noise dominates.

In Figure 1(c), the *nsr* is plotted for different values of ζ, which is the bid-ask spread. It is fairly evident that sampling at 10 seconds, introduces a large microstructure noise such that the variance of the signal is totally dominated by the noise term. For $\zeta = 0.001$ and $\gamma = 0.5$, the *nsr* is equal to 1.95, when $\Delta = 10$ seconds, and is 1.36 for $\Delta = 30$ minutes.

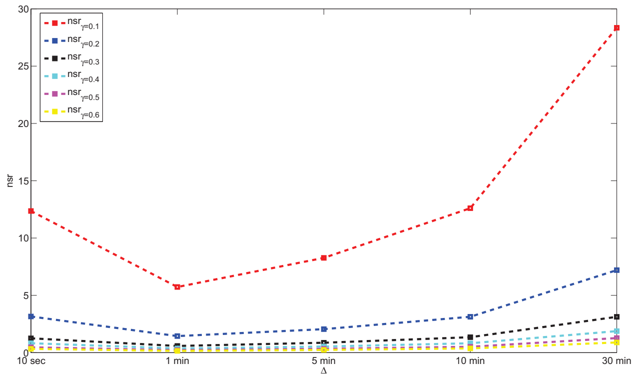
3.1.3 Stationary instantaneous volatility

For the stationary case we report the results corresponding to the following set of parameter values: $k = 0.9$, $\psi = -9.2$, $\gamma = \{0.5, 0.7\}$ and $d = 0.4$. The parameter ψ is the long-run mean of the log-volatility and $\psi = -9.2$ corresponds to an annualized volatility of approximately 16%. This combination of parameters generates a *nsr* which, in absence of microstructure noise, ranges between 2% when $\Delta = 1$ minute, to 96% when $\Delta = 30$ minutes. When Δ is 5 minutes, the *nsr* is between 10% ($\gamma = 0.7$) and 20% ($\gamma = 0.5$), and is consistent with the findings in Meddahi (2002).

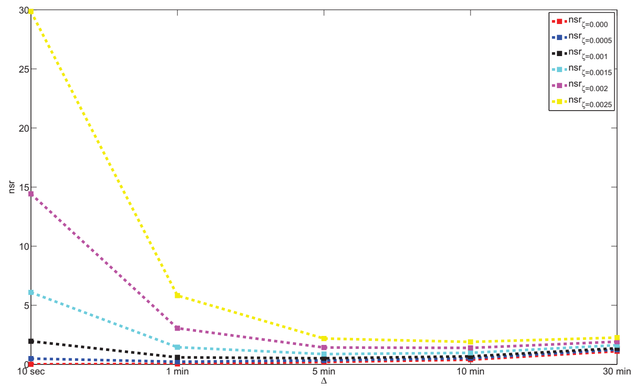
Table 1 reports the percentage bias and RMSE of the estimated long memory parameter when $d = 0.4$, obtained with the LW and the corrected LW estimators, see (21), for different choices of Δ, γ and ζ. In both panels, the estimates of d based on the IV_t are, as expected, the closest to the true value, and the percentage bias, smaller than 1%, is due to the Monte Carlo variance. However, in the real world, IV_t is unobservable and we rely on the realized measures to conduct inference on the degree of long memory. When $\zeta = 0$, the best LW estimates of d are obtained using all available returns, while the largest



(a) Noise-to-signal ratio, $\mathrm{Var}(\xi_t^{\Delta})/\,\mathrm{Var}(IV_t)$, as a function of $\Delta \in (10sec, 30min)$, with $\gamma = 0.5$ and $\zeta = 0$. Each line corresponds to a different value of d, i.e., $d = \{0, 0.1, 0.2, 0.3, 0.4, 0.45\}$.



(b) Noise-to-signal ratio, $\mathrm{Var}(\xi_t^{\Delta})/\,\mathrm{Var}(IV_t)$, as a function of $\Delta \in (10sec, 30min)$, with $\zeta = 0.001$ and $d = 0.4$. Each line corresponds to a different value of γ, i.e., $\gamma = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6\}$.



(c) Noise-to-signal ratio, $\mathrm{Var}(\xi_t^{\Delta})/\,\mathrm{Var}(IV_t)$, as function of $\Delta \in (10sec, 30min)$, with $\gamma = 0.5$ and $d = 0.4$. Each line corresponds to a different ζ, i.e., $\zeta = \{0, 0.0005, 0.001, 0.0015, 0.0020, 0.0025\}$.

Figure 1: Simulated noise-to-signal ratio.

Table 1: Bias and root mean squared error of Monte Carlo estimates of d

(a) $d = 0.4$ AND $\gamma = 0.5$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)
IV	-0.67	0.0546	-0.67	0.0565	-0.67	0.0546	-0.67	0.0546	-0.67	0.0546	-0.67	0.0572
RV^{all}	-0.74	0.0546	2.96	0.0571	-5.32	0.0597	-0.96	0.0570	-33.96	0.1508	-5.56	0.1319
RV^1	-1.17	0.0546	2.50	0.0465	-3.16	0.0559	0.82	0.0457	-13.22	0.0775	-0.94	0.0810
RV^5	-3.09	0.0558	0.73	0.0452	-4.50	0.0574	-0.45	0.0459	-9.37	0.0666	1.14	0.0719
RV^{10}	-5.45	0.0583	-0.25	0.0470	-6.65	0.0601	-0.98	0.0481	-10.39	0.0689	1.26	0.0753
RV^{30}	-13.98	0.0778	-1.28	0.0769	-15.03	0.0809	-1.61	0.0794	-18.08	0.0904	-0.24	0.0913

(b) $d = 0.4$ AND $\gamma = 0.7$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)
IV	-0.94	0.0490	-0.94	0.0490	-0.94	0.0490	-0.94	0.0490	-0.94	0.0490	-0.94	0.0490
RV^{all}	-1.02	0.0490	-0.82	0.0488	-4.10	0.0519	1.48	0.0583	-24.53	0.1172	-4.11	0.1081
RV^1	-1.35	0.0491	-0.17	0.0483	-2.54	0.0501	1.25	0.0521	-9.38	0.0638	-0.26	0.0735
RV^5	-2.98	0.0501	1.17	0.0525	-3.82	0.0511	1.41	0.0559	-7.22	0.0568	0.39	0.0667
RV^{10}	-4.59	0.0521	1.54	0.0585	-5.37	0.0534	1.31	0.0610	-8.19	0.0586	0.50	0.0692
RV^{30}	-11.29	0.0662	-0.41	0.0750	-11.98	0.0683	-0.66	0.0769	-13.93	0.0743	0.82	0.0813

Notes: $\hat{d}$ denotes the LW estimator of the long memory parameter, while $\hat{d}_c$ is the corrected LW estimator, see (21). The term *Bias* is referred to the relative percentage bias, defined in (26)-(27). The estimates are based on 1,000 samples of 2,500 daily observations from model (22)-(23) with parameter values indicated in table and discretization step set to 10 seconds. The bandwidth used in the estimation of d is $m = T^{0.65}$.

negative biases are those obtained with RV^{30}. The negative bias becomes larger as γ gets smaller, a result of the increase in the nsr ratio, displayed in Figure 1(b). This is coherent with the fact that the only source of noise in this case is the discretization error, so that increasing the latter, produces more biased estimates. When $\zeta = 0.001$ or $\zeta = 0.002$, the largest negative bias of LW is that obtained with RV^{all} (between -24% and -33%), while the bias of RV^{30} is between -13% and -18%. In presence of microstructure noise, the best LW estimates of d are obtained sampling at 1 and 5 minutes, and the bias is approximately -10%, so that the average of the Monte Carlo estimates is approximately 0.36. Interestingly, correcting for the presence of the measurement error improves the quality of the estimates, in terms of bias and RMSE, for any choice of Δ, and the relevance of the correction becomes evident as ζ increases. It is noteworthy that, despite the corrected LW estimator is asymptotically less efficient then the LW estimator, the RMSE of $\hat{d}_c$ is often smaller than that of $\hat{d}$. This means that the squared bias component of the RMSE prevails on the variance component, thus confirming the relevance of correcting for the measurement error.

3.1.4 Nonstationary instantaneous volatility

The simulated trajectories for the nonstationary $\sigma^2(t)$ are obtained setting $\gamma = 0.2$ and $d = \{-0.3, -0.4\}$. This implies that the fractional integration order, $\delta = 1 + d$, of the instantaneous and IV_t is 0.7 and 0.6, respectively. We initialize each simulated path with $p(0) = \ln(100)$ and $\sigma^2(0) = \exp(\psi) > 0$. Figure 2 reports a simulated trajectory of the IV_t under non-stationarity; it appears very realistic and, in particular, the model is able to generate long periods of high volatility which are typical of financial turmoils.

In the nonstationary setup, with $\delta = 0.6$, the nsr ranges between 10% when $\Delta = 1$ minute, to 300% when $\Delta = 30$ minutes. With $\delta = 0.7$, the nsr ranges between 3% when $\Delta = 1$ minute, to 82% when $\Delta = 30$ minutes. The reduction in the nsr obtained when $\delta = 0.7$ is due to higher persistence of the signal. The nsr corresponding to a Δ equal to 5 minutes, is 15%.

A similar evidence to that found in the stationary case emerges also when the $\sigma^2(t)$ is long memory but nonstationary, which could be the relevant case in practice, see Table 2. Firstly, the LW estimate, $\hat{\delta}$, based on IV_t is very close to the value $\delta = 1 + d$, which is the same as that of $\sigma^2(t)$. This is in accordance with Proposition 1, for long memory orders in the range $(0, 1)$. When $\zeta = 0$ and $\gamma = 0.2$, the variance of the noise dominates the signal as Δ increases, hence, the impact of the discretization error on the LW estimates of δ is very large. For example, when $\zeta = 0$ and $\delta = 0.6$, the bias of $\hat{\delta}$ based on RV^{30} is negative and larger than 30%, and larger than 18% when $\delta = 0.7$.

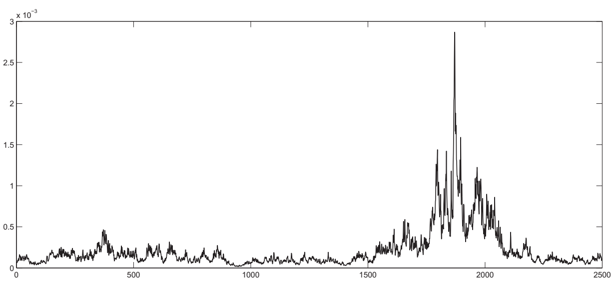


Figure 2: Nonstationary integrated variance. The figure plots a simulated trajectory of $IV_t = \int_{t-1}^{t} \sigma^2(s)\mathrm{d}s$ which is generated according to model (24), with $d = -0.3$ and $\gamma = 0.2$. $\sigma^2(0) = \exp(-9.2)$.

As expected, the negative bias increases, as ζ increases, and we observe extremely large negative biases for $\zeta = 0.002$, so that $\hat{\delta}$, based on the RV, falls in the stationary region, even though the IV_t is not stationary. For example, RV^5 has a negative bias equal to -28% when $\delta = 0.6$, meaning that $\hat{\delta} \approx 0.43$ on average. On the contrary, the corrected LW estimator, $\hat{\delta}_c$, provides unbiased estimates also in the nonstationary region, for both $\delta = 0.6$ and $\delta = 0.7$, and for all choices of Δ and ζ.

3.1.5 Leverage

Finally, the last set of simulations, reported in Table 3, investigates the impact on the estimates of d of the leverage effect, defined as the correlation between the innovation in the volatility process and that in the price process. For the stationary case the parameters are chosen as $k = 0.9$, $\psi = -9.2$, $\gamma = 0.5$, $d = 0.4$ and $\rho = -0.3$, where $\rho = corr(\mathrm{d}W(t), \mathrm{d}W_d(t))$, while for the nonstationary case,

Table 2: Bias and Root mean squared error of Monte Carlo estimates of $\delta = 1 + d$, when $\sigma^2(t)$ follows the process in (24)

(a) $\delta = 1 + d = 0.6$ AND $\gamma = 0.2$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)
IV	1.02	0.0596	1.02	0.0596	1.02	0.0596	1.02	0.0596	1.02	0.0596	1.02	0.0596
RV^{all}	0.62	0.0596	2.10	0.0501	-18.84	0.1339	1.78	0.1052	-60.73	0.3741	-2.56	0.1431
RV^{1}	-1.32	0.0601	3.29	0.0639	-10.55	0.0887	1.79	0.0901	-36.37	0.2316	0.82	0.1313
RV^{5}	-8.96	0.0797	2.38	0.0881	-14.38	0.1048	1.72	0.0970	-28.15	0.1810	1.55	0.1182
RV^{10}	-16.11	0.1137	1.68	0.1026	-20.10	0.1348	1.62	0.1079	-30.24	0.1927	1.81	0.1216
RV^{30}	-33.70	0.2105	1.23	0.1289	-35.61	0.2216	0.94	0.1317	-40.66	0.2514	0.87	0.1365

(b) $\delta = 1 + d = 0.7$ AND $\gamma = 0.2$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)
IV	-0.26	0.0639	-0.26	0.0639	-0.26	0.0639	-0.26	0.0639	-0.26	0.0639	-0.26	0.0639
RV^{all}	-0.41	0.0638	0.07	0.0571	-9.07	0.0999	0.18	0.0911	-37.06	0.2938	-2.28	0.1485
RV^{1}	-1.12	0.0638	0.20	0.0465	-4.66	0.0720	1.42	0.0827	-18.91	0.1644	-0.99	0.1104
RV^{5}	-4.40	0.0703	-0.26	0.0452	-6.73	0.0790	0.59	0.0888	-13.90	0.1213	-0.83	0.1015
RV^{10}	-7.70	0.0837	-0.99	0.0470	-9.74	0.0835	0.01	0.0957	-15.32	0.1272	-0.59	0.1073
RV^{30}	-18.57	0.1442	-2.70	0.0769	-19.80	0.1520	-1.02	0.1150	-23.04	0.1739	-1.16	0.1195

Notes: $\hat{\delta}$ denotes the LW estimator, while $\hat{\delta}_c$ is the corrected LW estimator, see (21). The term *Bias* is referred to the relative percentage bias, defined in equation (26)-(27). The estimates are based on 1,000 samples of 2,500 daily observations from model (22)-(23) with parameter values indicated in table and discretization step set to 10 seconds. The bandwidth used in the estimation of δ is $m = T^{0.65}$. When $d = -0.4$ and $d = -0.3$, then $\delta = 0.6$ and $\delta = 0.7$, respectively.

they are $\gamma = 0.2$, $d = -0.3$ and $\rho = -0.3$. The value $\rho = -0.3$ is chosen according to the findings in Andersen et al. (2002). Table 3 reports the simulation results for the case in which the innovations of the volatility process are correlated with the innovations of the price process (leverage effect). From Table 4(a) it emerges that, for intermediate choices of Δ, the corrected LW estimator is generally robust to the presence of correlation between the signal and the noise. In particular, the correction works better for intermediate choices of Δ. This indirectly confirms the finding in Meddahi (2002, p.493), that the correlation between the IV_t and the measurement error increases non-linearly with Δ, when $\rho \neq 0$. In the stationary case, the correlation is always positive and it ranges between 40% at 1 minute frequency and 60% at 30 minutes frequency. On the contrary, in the nonstationary case, the leverage effect produces a negative correlation that decreases with the sampling frequency, being -0.36% at 1 minute and -0.19% at 30 minutes. In this case, Table 4(a) shows that the leverage effect has a little impact on $\hat{\delta}_c$, which is well centered on the true parameter value also for large values of Δ, while the LW estimator performs as in the no-leverage case.

Summarizing, the simulation results highlight the effectiveness, in terms of bias reduction, of the corrected LW estimator of long memory of the IV_t, even when the volatility signal is nonstationary and in presence of leverage effect. More importantly, the corrected estimator is robust to the choice of the sampling frequency used for the computation of RV_t^{Δ}. A recent paper by Arteche (2012) proposes a modification of the Hurvich et al. (2005) estimator to account for the possible correlation between signal and noise, which may emerge under the presence of leverage. A detailed theoretical and empirical investigation of the consequences of the leverage on the estimation of long memory in IV is left for future research.

Table 3: Bias and Root mean squared error of Monte Carlo estimates of d and δ, under leverage effect

(a) $d = 0.4$, $\gamma = 0.5$ AND $\rho = -0.3$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)	Bias $\hat{d}$	RMSE($\hat{d}$)	Bias $\hat{d}_c$	RMSE($\hat{d}_c$)
IV	-0.91	0.0554	-0.91	0.0554	-0.91	0.0554	-0.91	0.0554	-0.91	0.0554	-0.91	0.0554
RV^{all}	-2.22	0.0568	3.21	0.0494	-6.77	0.0626	3.27	0.0735	-34.19	0.1540	-4.78	0.1417
RV^{1}	-7.50	0.0668	3.07	0.0763	-8.47	0.0685	2.44	0.0778	-13.83	0.0832	-1.10	0.0888
RV^{5}	-20.07	0.1081	-5.55	0.1085	-20.26	0.1087	-5.75	0.1090	-21.11	0.1122	-6.34	0.1106
RV^{10}	-26.95	0.1380	-11.35	0.1337	-27.04	0.1383	-11.32	0.1337	-27.30	0.1394	-11.56	0.1344
RV^{30}	-34.51	0.1682	-16.86	0.1610	-34.53	0.1683	-16.86	0.1609	-34.58	0.1686	-16.69	0.1611

(b) $\delta = 1 + d = 0.7$, $\gamma = 0.2$ AND $\rho = -0.3$.

	$\zeta = 0.000$				$\zeta = 0.001$				$\zeta = 0.002$			
	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)	Bias $\hat{\delta}$	RMSE($\hat{\delta}$)	Bias $\hat{\delta}_c$	RMSE($\hat{\delta}_c$)
IV	2.16	0.0352	2.16	0.0352	2.16	0.0352	2.16	0.0352	2.16	0.0352	2.16	0.0352
RV^{all}	-0.57	0.0652	3.99	0.0777	-9.13	0.1008	0.03	0.0932	-37.27	0.2942	-2.07	0.1611
RV^{1}	-1.21	0.0655	-0.71	0.0535	-5.18	0.0760	-2.12	0.0629	-19.91	0.1710	-0.81	0.1097
RV^{5}	-4.24	0.0711	1.15	0.0755	-6.76	0.0806	0.39	0.0833	-14.36	0.1257	-0.88	0.1024
RV^{10}	-7.60	0.0843	0.44	0.0887	-9.65	0.0941	-0.02	0.0931	-15.69	0.1313	-0.60	0.1013
RV^{30}	-18.18	0.1435	0.34	0.1102	-19.48	0.1514	0.19	0.1128	-23.11	0.1759	0.16	0.1163

Notes: $\hat{d}$ and $\hat{\delta}$ denote the LW estimators, while $\hat{d}_c$ and $\hat{\delta}_c$ are the corrected LW estimators, see (21). The term *Bias* is referred to the relative percentage bias, defined in equation (26)-(27). The estimates are based on 1,000 samples of 2,500 daily observations from models (22), (23) and (24) with parameter values indicated in table and discretization step set to 10 seconds. The bandwidth used in the estimation of d and δ is $m = T^{0.65}$. In Panel (a), the IV process is stationary, with $d = 0.4$. In Panel (b), the IV process is nonstationary, with $\delta = 0.7$.

4 Empirical Analysis

We estimate the long memory of IV_t, based on the RV series of 28 stocks traded on NYSE. The sample period ranges from January 2, 2001 to December 31, 2007, for a total of 1760 trading days.[11] The RV series are computed with alternative sample frequencies, say 1 minute, 5 minutes, 10 minutes, 15 minutes and 30 minutes.

Table 4 reports the estimates of the long memory parameter d. The corrected estimates signal that volatilities are generated by a nonstationary long memory process. Firstly, on average, the estimates obtained correcting for the measurement error are higher than those obtained with the LW estimator, which lies in the stationary region. Secondly, the corrected estimates are relatively constant with respect to the choice of the sampling frequency used for the computation of the RV. Instead, the non corrected ones are characterized by a downward trend with respect to Δ. The dispersion of the corrected estimates, as measured by $\sigma(\hat{d})$ and $\hat{d}_1 - \hat{d}_{30}$, is smaller than that observed for the LW estimates. This evidence is clear from a visual inspection of Figure 3, which reports the *long memory signature plot* for four stocks in the sample. The LW estimates based on the RV series (dashed line) fall in most cases in the stationary region, while it is evident the downward trend with respect to Δ. On the other hand, the corrected estimates of d (solid line) are always above the Whittle estimates and are constant across different choices of Δ, in line with the simulation results. We think that it is important to stress the fact that the corrected LW estimator always lies in the nonstationary region, suggesting that the volatility process could be a nonstationary process. From this point of view, the fact that the LW estimates of d, based on RV_t^{Δ}, turn out to be less than 0.5, namely a stationary long memory process, is mainly due to the role of the measurement error.

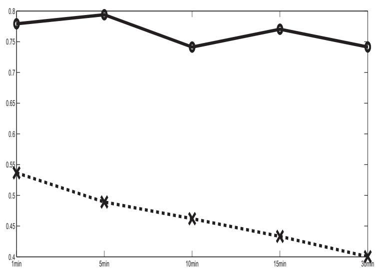
This means that using a biased long memory estimator leads to wrong conclusions on the stationarity of the integrated and instantaneous volatility processes. Using a similar argument, but in a discrete-time domain framework, Hansen and

[11] We avoid the possible upward bias in the semiparametric estimates of d, due to the presence of large shifts as generated by changing bull and bear markets, during the 2008-2009 financial crisis.

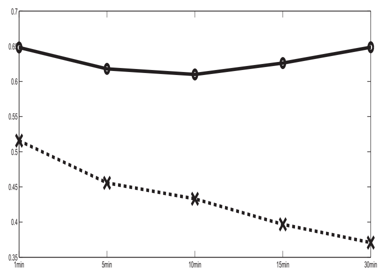
Table 4: Long memory estimates based on RV_t^{Δ} of 28 NYSE stocks

	Local Whittle								Corrected Local Whittle							
Δ	$1min$	$5min$	$10min$	$15min$	$30min$	$\bar{d}$	$\sigma(\hat{d})$	$\hat{d}_1 - \hat{d}_{30}$	$1min$	$5min$	$10min$	$15min$	$30min$	$\bar{d}$	$\sigma(\hat{d})$	$\hat{d}_1 - \hat{d}_{30}$
AXP	0.5722	0.5299	0.5039	0.4718	0.4146	0.4985	0.0596	0.1576	0.6543	0.6257	0.5937	0.6043	0.6665	0.6289	0.0313	-0.0122
BA	0.5378	0.4954	0.4316	0.4129	0.3523	0.4460	0.0724	0.1855	0.6281	0.6015	0.5893	0.5752	0.5439	0.5876	0.0312	0.0842
C	0.5465	0.5172	0.4814	0.4583	0.4414	0.4890	0.0429	0.1051	0.5851	0.5893	0.5731	0.5439	0.5847	0.5752	0.0185	0.0004
CAT	0.4690	0.4363	0.4109	0.4003	0.3429	0.4119	0.0468	0.1261	0.6546	0.6087	0.5857	0.5404	0.5893	0.5957	0.0413	0.0653
DD	0.5637	0.5177	0.4803	0.4452	0.3955	0.4805	0.0648	0.1682	0.6139	0.6207	0.6565	0.6315	0.6175	0.6280	0.0172	-0.0036
EMR	0.5245	0.4844	0.4615	0.4380	0.3852	0.4587	0.0520	0.1393	0.5962	0.6358	0.6339	0.6303	0.6275	0.6247	0.0163	-0.0313
F	0.4820	0.4276	0.3966	0.3672	0.3344	0.4016	0.0567	0.1476	0.6803	0.6002	0.5951	0.5687	0.5646	0.6018	0.0466	0.1157
FDX	0.4485	0.4389	0.4057	0.3687	0.3014	0.3926	0.0599	0.1471	0.5405	0.5181	0.5165	0.5142	0.5339	0.5246	0.0118	0.0066
GE	0.5291	0.4799	0.4589	0.4186	0.3892	0.4551	0.0543	0.1399	0.6481	0.5601	0.5550	0.5625	0.5790	0.5809	0.0386	0.0691
GS	0.5157	0.4557	0.4331	0.3967	0.3708	0.4344	0.0560	0.1449	0.6483	0.6179	0.6099	0.6261	0.6486	0.6302	0.0177	-0.0003
HD	0.5619	0.5187	0.4775	0.4497	0.4233	0.4862	0.0551	0.1386	0.7000	0.6809	0.6522	0.6606	0.6250	0.6637	0.0285	0.0750
HON	0.4636	0.4458	0.4001	0.3801	0.3592	0.4098	0.0440	0.1044	0.5513	0.5372	0.5513	0.5470	0.5091	0.5392	0.0178	0.0422
IBM	0.5278	0.4825	0.4471	0.4343	0.4145	0.4612	0.0447	0.1133	0.6694	0.6396	0.6131	0.6058	0.5946	0.6245	0.0301	0.0748
JNJ	0.5615	0.5125	0.4812	0.4507	0.4222	0.4856	0.0542	0.1393	0.7323	0.7028	0.6519	0.6292	0.6430	0.6718	0.0438	0.0893
MCD	0.4934	0.4259	0.3876	0.3571	0.3294	0.3987	0.0640	0.1640	0.6087	0.5459	0.5875	0.5623	0.6894	0.5988	0.0560	-0.0807
MET	0.4824	0.4581	0.4199	0.3878	0.3483	0.4193	0.0537	0.1341	0.7013	0.6819	0.6791	0.6402	0.6183	0.6642	0.0339	0.0830
NEM	0.5049	0.4360	0.4070	0.3593	0.3318	0.4078	0.0677	0.1731	0.6699	0.5947	0.5493	0.5988	0.5569	0.5939	0.0479	0.1130
PEP	0.5740	0.4953	0.4653	0.4369	0.3992	0.4741	0.0661	0.1748	0.6806	0.6908	0.6811	0.6767	0.6356	0.6730	0.0215	0.0450
PFE	0.4907	0.4313	0.3917	0.3729	0.3255	0.4024	0.0623	0.1652	0.5596	0.5278	0.5405	0.5451	0.5416	0.5429	0.0114	0.0180
PG	0.5616	0.5052	0.4811	0.4469	0.4024	0.4794	0.0600	0.1592	0.7035	0.6450	0.5960	0.5718	0.6184	0.6269	0.0506	0.0851
T	0.5590	0.4747	0.4347	0.4119	0.3739	0.4508	0.0706	0.1851	0.5590	0.5195	0.5212	0.5231	0.5574	0.5360	0.0203	0.0016
TWX	0.4630	0.4465	0.4551	0.4392	0.4274	0.4462	0.0138	0.0356	0.5455	0.5232	0.5204	0.5065	0.4989	0.5189	0.0179	0.0466
TXN	0.5220	0.4925	0.4778	0.4398	0.3963	0.4657	0.0488	0.1257	0.6851	0.6469	0.6060	0.5927	0.6293	0.6320	0.0363	0.0558
UPS	0.4378	0.3986	0.3754	0.3606	0.3042	0.3753	0.0493	0.1336	0.5236	0.5433	0.5237	0.5146	0.4886	0.5188	0.0199	0.0350
VZ	0.5366	0.4891	0.4620	0.4332	0.4002	0.4642	0.0523	0.1364	0.7792	0.7940	0.7413	0.7704	0.7415	0.7653	0.0234	0.0377
WFC	0.5030	0.4554	0.4325	0.4200	0.3855	0.4393	0.0437	0.1175	0.7248	0.6689	0.6096	0.6336	0.6508	0.6575	0.0435	0.0740
WMT	0.5743	0.5305	0.5028	0.4784	0.4138	0.5000	0.0599	0.1605	0.6436	0.6226	0.6212	0.5975	0.6058	0.6181	0.0177	0.0378
XOM	0.5292	0.4898	0.4481	0.4354	0.3634	0.4532	0.0623	0.1658	0.7603	0.6407	0.6311	0.5862	0.6031	0.6443	0.0684	0.1572
AVG	0.5191	0.4740	0.4432	0.4169	0.3767	0.4460	0.0549	0.1424	0.6445	0.6137	0.5995	0.5914	0.5987	0.6096	0.0307	0.0459

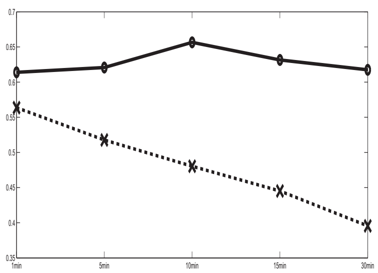
Notes: The names of the companies corresponding to the stock tickers reported in Table can be found at `http://www.nyse.com`. The RV_t^{Δ} is computed with different choices for Δ, that is 1, 5, 10, 15 and 30 minutes. Table reports the estimates of d obtained using the LW estimator and the corrected LW, see (20). The bandwidth used for the estimation is $m = T^{0.8}$. $\bar{d}$ is the average of the estimates, with respect to the different choices of Δ. $\sigma(\hat{d})$ is the sample standard deviation of the estimates of d computed with different RV_t^{Δ}. $\hat{d}_1 - \hat{d}_{30}$ is the difference between the estimates obtained with the RV at 1 minute and 30 minutes. AVG is the average for each column.



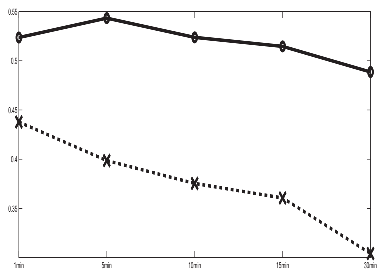
(a) RV OF C



(b) RV OF GS



(c) RV OF DD



(d) RV OF UPS

Figure 3: *Long memory signature plots*: Long memory parameter estimates for different sampling frequencies (1, 5, 10, 15 and 30 minutes). Dashed lines represent the LW estimator of the memory parameter (obtained minimizing the function in (20) concentrated with respect to G with $\beta = 0$). Solid lines represent the corrected LW estimator (see (21)).)

Lunde (2010) have proposed an instrumental variable estimator of the persistence of the signal when the latter is a unit root process. To the best of our knowledge, the consequences of a fractional, but nonstationary, volatility process are not studied yet in the literature and the evidence reported here deserves a more detailed analysis.

5 Conclusions

A stylized fact is that RV has long memory. In this paper, we investigate the dynamic properties and the source of the long-range dependence of RV. First, we find that, when the instantaneous volatility is driven by a fractional Brownian motion, the IV_t is characterized by the same degree of long-range dependence, d. As a consequence, the RV inherits this property, since the spectral density of RV is equal to the spectral density of IV, plus a term which depends on the variance of the measurement error.

The additional term in the spectral density of RV impacts on the finite sample properties of the semiparametric estimates of d, since they crucially depend on the use of RV in place of the unobservable IV. In absence of microstructure noise, the RV spectral density converges to the spectral density of IV, as $\Delta \to 0$. When the presence of microstructure noise prevents us from using all the available price observations, the additional component in the spectral density, which depends on the discretization error and on the microstructure noise, significantly affects the finite sample bias of semiparametric estimates of d.

We adopt a correction of the LW estimator along the lines of Hurvich et al. (2005). A Monte Carlo experiment confirms that the correction of the local Whittle estimator is robust to the measurement error for all choices of Δ. Thus the trade-off between discretization error and microstructure noise is neutralized by adopting a corrected version of the LW estimator. Finally, the estimation of the long memory of 28 NYSE stocks emphasizes the practical importance of considering the measurement error when estimating the degree of long memory of IV.

The corrected estimates of d suggest that the IV and the instantaneous volatility can be nonstationary processes. In this study we have not considered the role of jumps in prices and their potential effect on the estimation of long memory in IV. This is left for future research.

A Proofs

A.1 Proof of Proposition 1

Given that $IV_t = \int_{t-1}^{t} \tilde{\sigma}^2(s)ds$. Following Chambers (1996) we express the integral operator in the definition of IV as a simple filter that has transfer function

$$T(\lambda) = \int_0^1 e^{-i\lambda u} du = \frac{1}{(-i\lambda)}[e^{-i\lambda} - 1].$$

Therefore the spectral density (or pseudo spectral density) of IV is given by

$$f_{IV}(\lambda) = |T(\lambda)|^2 f_{\sigma^2}(\lambda). \tag{28}$$

The limit of $f_{IV}(\lambda)$ for $\lambda \to 0$ is

$$\lim_{\lambda \to 0} f_{IV}(\lambda) = \lim_{\lambda \to 0} [|T(\lambda)|^2 f_{\sigma^2}(\lambda)] \tag{29}$$

Since $|T(\lambda)|^2 = \frac{2(1-\cos(\lambda))}{|\lambda|^2}$ and $(1-\cos(\lambda)) \approx |\lambda|^2/2$ as $\lambda \to 0$, then $\lim_{\lambda \to 0} |T(\lambda)|^2 = 1$. Thus,

$$\lim_{\lambda \to 0} \lambda^{2d} f_{IV}(\lambda) = \lim_{\lambda \to 0} \lambda^{2d} f_{\sigma^2}(\lambda) = c, \tag{30}$$

that is IV has the same degree of long memory of $\sigma^2(t)$, which is equivalent to $\sigma^2(t)$.

A.2 Proof of Proposition 2

Assume that the processes for $p(t)$ and RV_t^{Δ} are those in (1) and (6). Assume also that $\sigma^2(t)$ is such that condition (2) is verified, and that $m(t) = \mu$ and no leverage effect, then

$$u_{t,i}^{\Delta} \stackrel{\mathcal{L}}{=} \sigma_{t,i,\Delta}^2 \left(z_{t,i}^2 - 1\right) + \Delta^2\mu^2 + 2\Delta\mu\sigma_{t,i,\Delta} z_{t,i}. \tag{31}$$

It is easy to show that

(i) $E(u_t^{\Delta}) = \Delta\mu^2$;

(ii) $\mathrm{Var}(u_t^{\Delta}) = 2\Delta^{-1}E\left[(\sigma^2_{t,i,\Delta})^2\right] + 4\Delta\mu^2 E\left[\sigma^2_{t,i,\Delta}\right]$;

(iii) u_t^{Δ} is dynamically uncorrelated, i.e., $\mathrm{Cov}(u_t^{\Delta}, u_{t+h}^{\Delta}) = 0$, for any integer $h \neq 0$;

(iv) The error term u_t^{Δ} is uncorrelated with IV_t;

(v) $\mathrm{Cov}(RV_t^{\Delta}, RV_{t-h}^{\Delta}) = \mathrm{Cov}(IV_t, IV_{t-h})$, for any integer $h \neq 0$.

Thus, for $\Delta > 0$ and $0 < d < 1/2$ the spectral density of RV_t^{Δ} is given by

$$
\begin{aligned}
f_{RV^{\Delta}}(\lambda) &= \frac{1}{2\pi}\left\{\mathrm{Var}(IV_t) + \mathrm{Var}(u_t^{\Delta}) + 2\sum_{j=1}^{\infty}\left[\mathrm{Cov}(IV_t, IV_{t-j})\cos(\lambda j)\right]\right\} \\
&= f_{IV}(\lambda) + f_{u^{\Delta}}(\lambda) \qquad (32)
\end{aligned}
$$

When $\Delta > 0$ and $1/2 \leq d < 1$, the pseudo spectral density of RV_t^{Δ} is given by the expectation of its sample periodogram, i.e.

$$
\begin{aligned}
f_{RV^{\Delta}}(\lambda) \equiv E(I_{RV^{\Delta}}(\lambda)) &= E(I_{IV+u^{\Delta}}(\lambda)) \\
&= E(I_{IV}(\lambda)) + E(I_{u^{\Delta}}(\lambda)) \\
&= f_{IV}(\lambda) + f_{u^{\Delta}}(\lambda) \qquad (33)
\end{aligned}
$$

where $I.(\lambda)$ is the sample periodogram.

Therefore, for $0 < d < 1$, $\lim_{\lambda\to 0}\lambda^{2d} f_{RV^{\Delta}}(\lambda) = \lim_{\lambda\to 0}\lambda^{2d}(f_{IV}(\lambda) + f_{u^{\Delta}}(\lambda)) = c$ with $c > 0$, where $f_{u^{\Delta}}(\lambda) = \frac{\mathrm{Var}(u_t^{\Delta})}{2\pi}$.

Given that $\mathrm{Var}(u_t^{\Delta})$ converges to zero as $\Delta \to 0$, so that $f_{u^{\Delta}}(\lambda) \to 0$. This implies that

$$
\lim_{\Delta\to 0} f_{RV^{\Delta}}(\lambda) = f_{IV}(\lambda) \qquad (34)
$$

the proof then follows from Proposition 1, and multiplying both sides by λ^{2d}, letting $\lambda \to 0$.

A.3 Proof of Proposition 3

Consider the processes $\tilde{p}(t)$, RV_t^{Δ}, ξ_t^{Δ} defined respectively in (14), (6) and (16). Let $m(t) = \mu$ and assume no leverage effect. Assume also that $\sigma^2(t)$ is such that condition (2) is verified. First, in order to characterize the spectral density of RV_t^Δ, we need to obtain the moments of the measurement error, $\xi_t^\Delta = \sum_{i=1}^n \xi_{t,i}^\Delta$.

(a) $\xi_{t,i}^\Delta$ is defined as

$$\xi_{t,i}^\Delta \overset{\mathcal{L}}{=} \sigma_{t,i,\Delta}^2 \left(z_{t,i}^2 - 1\right) + \Delta^2\mu^2 + \eta_{t,i,\Delta}^2 + 2\left(\sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right) + 2\Delta\mu\eta_{t,i,\Delta} + 2\Delta\mu\sigma_{t,i,\Delta} z_{t,i}, \tag{35}$$

hence

$$\begin{aligned} E\left(\xi_{t,i}^\Delta\right) &= E\left[\sigma_{t,i,\Delta}^2\left(z_{t,i}^2-1\right)\right] + E\left(\Delta^2\mu^2\right) + E\left(\eta_{t,i,\Delta}^2\right) \\ &\quad + 2E\left(\sigma_{t,i,\Delta} z_{t,i}\eta_{t,i,\Delta}\right) + 2\Delta\mu E\left(\eta_{t,i,\Delta}\right) + 2\Delta\mu E\left(\sigma_{t,i,\Delta} z_{t,i}\right) \\ &= \Delta^2\mu^2 + E\left(\eta_{t,i,\Delta}^2\right) \\ &= \Delta^2\mu^2 + \sigma_\eta^2, \end{aligned}$$

where $\sigma_\eta^2 = \mathrm{Var}[\eta_{t,i,\Delta}] = 2\,\mathrm{Var}[\epsilon_{t,i,\Delta}]$. Because $\sigma_{t,i,\Delta}$, $z_{t,i}$, and $\eta_{t,i,\Delta}$ are mutually independent, $E\left(\sigma_{t,i,\Delta} z_{t,i}\eta_{t,i,\Delta}\right) = E(\sigma_{t,i,\Delta}) \cdot E(z_{t,i}) \cdot E(\eta_{t,i,\Delta}) = 0$. It follows that $E\left(\sum_{i=1}^n \xi_{t,i}^\Delta\right) = \Delta^{-1}\sigma_\eta^2 + \Delta\mu^2$;

(b) The covariance between $\xi_{t,i}^\Delta$ and $\xi_{t,j}^\Delta$ can be written as

$$\begin{aligned} \mathrm{Cov}\left(\xi_{t,i}^\Delta, \xi_{t,j}^\Delta\right) &= E\left[u_{t,i}^\Delta u_{t,j}^\Delta\right] + E\left[u_{t,i}^\Delta \eta_{t,j,\Delta}^2\right] + 2E\left[u_{t,i}^\Delta\left(\sigma_{t,j,\Delta} z_{t,j}\eta_{t,j,\Delta}\right)\right] \\ &\quad + 2\Delta\mu E\left[u_{t,i}^\Delta \eta_{t,j,\Delta}\right] + 4\Delta\mu\left[\left(\sigma_{t,i,\Delta} z_{t,i}\eta_{t,i,\Delta}\right)\eta_j\right] \\ &\quad + E\left[\eta_{t,i,\Delta}^2 u_{t,j}^\Delta\right] + E\left[\eta_{t,i,\Delta}^2\eta_{t,j,\Delta}^2\right] + 2E\left[\eta_{t,i,\Delta}^2\left(\sigma_{t,j,\Delta} z_{t,j}\eta_{t,j,\Delta}\right)\right] \\ &\quad + 2E\left[\left(\sigma_{t,i,\Delta} z_{t,i}\eta_{t,i,\Delta}\right) u_{t,j}^\Delta\right] + 2E\left[\left(\sigma_{t,i,\Delta} z_{t,i}\eta_{t,i,\Delta}\right)\eta_{t,j}^2\right] \\ &\quad + 4E\left[\left(\sigma_{t,i,\Delta} z_{t,i}\eta_{t,i,\Delta}\right)\left(\sigma_{t,j,\Delta} z_{t,j}\eta_{t,j,\Delta}\right)\right] \\ &\quad + 2\Delta\mu\left[\eta_{t,i,\Delta}\eta_{t,j,\Delta}^2\right] + 2\Delta\mu\left[\eta_{t,i,\Delta}^2\eta_{t,j,\Delta}\right] + 2\Delta\mu\left[\eta_{t,i,\Delta}\left(\sigma_{t,j,\Delta} z_{t,j}\eta_{t,j,\Delta}\right)\right] \\ &\quad + 2\Delta\mu E\left[\eta_{t,i,\Delta} u_{t,j}^\Delta\right] + 4\Delta^2\mu^2 E\left[\eta_{t,i,\Delta}\eta_{t,j,\Delta}\right] - \sigma_\eta^4 - 2\Delta^2\mu^2\sigma_\eta^2 - \Delta^4\mu^4 \\ &= \sigma_\eta^4 + 2\Delta^2\mu^2\sigma_\eta^2 + \Delta^4\mu^4 - \sigma_\eta^4 - 2\Delta^2\mu^2\sigma_\eta^2 - \Delta^4\mu^4 = 0 \quad \forall i \neq j \end{aligned}$$

The covariance of ξ_t^Δ and ξ_{t+h}^Δ is equal to

$$\text{Cov}\left(\sum_{i=1}^{n}\xi_{t,i}^\Delta, \sum_{j=1}^{n}\xi_{t+h,j}^\Delta\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\text{Cov}\left(\xi_{t,i}^\Delta, \xi_{t+h,j}^\Delta\right) = 2n^2 \cdot 0 = 0 \quad \text{for any integer } h \neq 0. \tag{36}$$

(c) The variance of $\xi_{t,i}^\Delta$ is,

$$\begin{aligned}
\text{Var}\left(\xi_{t,i}^\Delta\right) &= \text{Var}(u_{t,i}^\Delta) + \text{Var}(\eta_{t,i}^2) + 4\,\text{Var}\left(\sigma_{t,i,\Delta} z_{t,i} \eta_{t,i,\Delta}\right) + 4\Delta^2\mu^2\,\text{Var}(\eta_{t,i}) \\
&= 2E\left[(\sigma_{t,i,\Delta}^2)^2\right] + 4\Delta^2\mu^2 E\left[\sigma_{t,i,\Delta}^2\right] + E(\eta_{t,i,\Delta}^4) - \sigma_\eta^4 + 4\sigma_\eta^2 E\left[\sigma_{t,i,\Delta}^2\right] + 4\Delta^2\mu^2\sigma_\eta^2,
\end{aligned}$$

it follows that the variance of ξ_t^Δ is

$$\begin{aligned}
\text{Var}\left(\xi_t^\Delta\right) &= \text{Var}\left(\sum_{i=1}^{n}\xi_{t,i}^\Delta\right) = \sum_{i=1}^{n}\text{Var}\left(\xi_{t,i}^\Delta\right) \\
&= 2\Delta^{-1}E\left[(\sigma_{t,i,\Delta}^2)^2\right] + 4\Delta\mu^2 E\left[(\sigma_{t,i,\Delta}^2)^2\right] + \Delta^{-1}(E(\eta_{t,i,\Delta}^4) - \sigma_\eta^4) \\
&\quad + 4\Delta^{-1}\sigma_\eta^2 E\left[\sigma_{t,i,\Delta}^2\right] + 4\Delta\mu^2\sigma_\eta^2
\end{aligned} \tag{37}$$

i. For $\Delta > 0$ and $0 < d < 1/2$, the spectral density of RV_t^Δ is therefore given by:

$$\begin{aligned}
f_{RV^\Delta}(\lambda) &= \frac{1}{2\pi}\left\{\text{Var}(IV_t) + \text{Var}(\xi_t^\Delta) + 2\sum_{j=1}^{\infty}\left[\text{Cov}(IV_t, IV_{t-j})\cos(\lambda j)\right]\right\} \\
&= f_{IV}(\lambda) + f_{\xi^\Delta}(\lambda).
\end{aligned}$$

When $\Delta > 0$ and $1/2 \leq d < 1$, the pseudo spectral density of RV_t^Δ is given by the expectation of the sample periodogram of RV, i.e.

$$\begin{aligned}
f_{RV^\Delta}(\lambda) \equiv E(I_{RV^\Delta}(\lambda)) &= E(I_{IV+\xi^\Delta}(\lambda)) \\
&= E(I_{IV}(\lambda)) + E(I_{\xi^\Delta}(\lambda)) \\
&= f_{IV}(\lambda) + f_{\xi^\Delta}(\lambda)
\end{aligned} \tag{38}$$

The proof then follows from Proposition 1, and multiplying both sides by λ^{2d}, letting $\lambda \to 0$.

ii. It is evident that when $\Delta \to 0$, $\text{Var}\left(\xi_t^\Delta\right) \to \infty$, so that $f_{RV^\Delta}(\lambda) \to \infty\ \forall\lambda$.

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