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Constraint qualifications for programming problems with axiomatic directional derivatives

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Constraint qualifications for programming problems with axiomatic directional derivatives

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Abstract

We give an overview of constraint qualifications and optimality conditions for a nonlinear programming problems where the functions involved have directional derivatives defined in an axiomatic way. We consider mainly the approach of Elster and Thierfelder (1985, 1988a, 1988b) and the approach of Giannessi (1989).

Key words

Constraint qualifications, optimality conditions, axiomatic directional derivatives, nonsmooth programming.

1. Introduction

Among the axiomatic approaches considered to unify the various definitions of generalized directional derivative for a function of several variables, the constructions proposed by Elster and Thierfelder (1985, 1988a, 1988b) and by Giannessi (1989) are perhaps the most known. The first approach is based on the concept of K-directional derivative, the second approach is based on the concept of G-derivative and G-semiderivative.

In the present paper we shall be concerned with some constraint qualifications for the mathematical programming problem

$$(\mathcal{P}) \qquad \min f(x), \quad g_i(x) \leq 0, \quad i \in I = \{1, 2, ..., m\},$$

where $f,\ g_i : \mathbb{R}^n \to \mathbb{R}$, are functions whose generalized (directional) differentiability is defined in an axiomatic way.

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Obviously, extensions to topological spaces are possible, as shown by the authors previously quoted.

The paper is organized as follows.

In section 2 we shall be concerned with the axiomatic approach of Elster and Thierfelder and on constraint qualifications based on this approach.

In section 3 we shall be concerned with the axiomatic approach of Giannessi and on constraint qualifications based on this approach.

In section 4 we make some comparisons between the two approaches considered and make some other remarks on a third axiomatic approach, proposed by Komlosi (1993) and by Komlosi and Pappalardo (1994).

2. Constraint qualifications with the κ-directional derivatives of Elster and Thierfelder

The various local cone approximations used in optimization theory, beginning from the works of Abadie (1967), Arrow, Hurwicz and Uzawa (1961), Hestenes (1966,1975), Kuhn and Tucker (1951) and many others, have induced Elster and Thierfelder (1985, 1988a, 1988b) to propose the following axiomatic definition of a first-order local cone approximation of a set $M \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$.

Definition 1

A map $K : 2^{\mathbb{R}^n} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is called a *local cone approximation* if for any set $M \subset \mathbb{R}^n$ and any point $x \in \mathbb{R}^n$ a cone $K(M, x)$ is associated, verifying the following axioms:

(a) $K(M, x) = K(M - x, 0)$;

(b) $K(M, x) = K(M \cap N(x))$, $\forall$ neighborhood $N(x)$;

(c) $K(M, x) = \emptyset, \quad \forall x \notin \text{cl}(M)$;

(d) $K(M, x) = \mathbb{R}^n, \quad \forall x \in \text{int}(M)$;

(e) $K(\varphi(M), \varphi(x)) = \varphi(K(M, x))$ for any linear homeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$;

(f) $0^+M \subset 0^+K(M, x)$, where $0^+M = \{d \in \mathbb{R}^n : x + \lambda d \in M, \forall \lambda > 0, \forall x \in M\}$ is the *recession cone* of the set M. It is assumed that $0^+\emptyset = \mathbb{R}^n$.

Elster and Thierfelder (1988a) have proved that the axioms a)-f) are independent; moreover, they prove that if $K(M, x)$ and $K_i(M, x)$, $i \in I = \{1, 2, ..., m\}$, are local cone approximations, then also $\text{int}(K(M, x))$, $\text{cl}(K(M, x))$, $\text{conv}(K(M, x))$, $\mathbb{R}^n \setminus [K(\mathbb{R}^n \setminus M, x)]$, $\bigcup_{i \in I} K_i(M, x)$, $\bigcap_{i \in I} K_i(M, x)$, $\sum_{i \in I} K_i(M, x)$, are local cone approximations.

The class of local cone approximations, defined by the axioms (a)-(f) is nonempty and, in particular, contains the following cones, often used in optimization theory.

Definition 2

Let $M \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

- The set

  $T(M,x) = \{y \in \mathbb{R}^n : \forall N(y), \forall \lambda > 0, \exists y' \in N(y), \exists t \in (0,\lambda) \text{ such that } x + ty' \in M\}$

  is the *Bouligand tangent cone* or *contingent cone*.

We recall that the closure of its convex hull is called *pseudotangent cone* (Guignard (1969)) and denoted by $P(M,x)$:

$$P(M,x) = \text{cl}\,(\text{conv}(T(M,x)))\,.$$

This cone is important in establishing the "weakest" constraint qualifications for the problem $(\mathcal{P})$. See Gould and Tolle (1971).

- The set

  $A(M,x) = \{y \in \mathbb{R}^n : \forall N(y), \exists \lambda > 0, \forall t \in (0,\lambda), \exists y' \in N(y) \text{ such that } x + ty' \in M\}$

  is the cone of the *attainable directions* or *Kuhn-Tucker tangent cone* or *Ursescu tangent cone*.

- The set

  $I(M,x) = \{y \in \mathbb{R}^n : \exists N(y), \exists \lambda > 0, \text{ such that } \forall t \in (0,\lambda), \forall y' \in N(y), x + ty' \in M\}$

  is the *cone of the interior directions*.

- The set

  $Q(M,x) = \{y \in \mathbb{R}^n : \exists N(y), \forall \lambda > 0, \exists t \in (0,\lambda) \text{ such that } \forall y' \in N(y), x + ty' \in M\}$

  is the *cone of the quasiinterior directions*.

- The set

  $Z(M,x) = \{y \in \mathbb{R}^n : \exists \lambda > 0 \text{ such that } \forall t \in (0,\lambda), x + ty \in M\}$

  is the *cone of the feasible directions*.

- The set

  $F(M,x) = \{y \in \mathbb{R}^n : \forall \lambda > 0, \exists t \in (0,\lambda) \text{ such that } x + ty \in M\}$

  is the *radial tangent cone* or *cone of the weakly feasible directions*.

- The set

  $$\begin{aligned} TC(M,x) = \; & \{y \in \mathbb{R}^n : \forall N(y), \exists \lambda > 0, \exists V(x) \text{ such that} \\ & \forall x' \in V(x) \cap M \cup \{x\}, \forall t \in (0,\lambda), \exists y' \in N(y) \text{ with } x' + ty' \in M\} \end{aligned}$$

  is the *Clarke tangent cone*.

- The set

  $$\begin{aligned} H(M,x) = \; & \{y \in \mathbb{R}^n : \exists V(x), \exists \lambda > 0, \text{ such that} \\ & \forall x' \in V(x) \cap M \cup \{x\}, \forall t \in (0,\lambda), \; x' + ty \in M\} \end{aligned}$$

  is the *hypertangent cone* (*of Rockafellar*).

∘ The set

$$E(M,x) = \{y \in \mathbb{R}^n : \exists U(y), \exists V(x), \exists \lambda > 0, \text{ such that } \forall x' \in V(x) \cap M \cup \{x\}, \forall t \in (0,\lambda), \forall y' \in U(y) \text{ we have } x' + ty' \in M\}$$

is the *cone of epilipschitzian directions.*

Remark 1

The definitions of the cones $TC(M,x)$, $H(M,x)$ and $E(M,x)$ are slightly different from the original definitions of Clarke (1983) an Rockafellar (1980, 1981), where $x' \in V(x) \cap M$. The requirement $x' \in V(x) \cap M \cup \{x\}$ allows to verify the (c) axiom of Definition 1. Giorgi and Guerraggio (1992) have proved that if $x \in \mathrm{cl}(M)$, the two types of definitions coincide.

By means of the concept of local cone approximation, Elster and Thierfelder (1988a, 1988b) give the following definition of generalized (axiomatic) directional derivative.

Definition 3

Let $f : \mathbb{R}^n \to [-\infty, +\infty]$, $x \in \mathbb{R}^n$, such that $|f(x)| < +\infty$ and a local cone approximation $K : 2^{\mathbb{R}^n \times \mathbb{R}} \times \mathbb{R}^n \times \mathbb{R} \to 2^{\mathbb{R}^n \times \mathbb{R}}$. The function $f^K(x,y) : \mathbb{R}^n \to [-\infty, +\infty]$ defined as

$$f^K(x,y) = \inf\left\{\beta \in \mathbb{R} : (y,\beta) \in K\left(\mathrm{epi} f,\ (x, f(x))\right)\right\},\ y \in \mathbb{R}^n,$$

is the *K-derivative* of f at x. For convention $\inf \emptyset = +\infty$.

It is easy to see that f^K is a positively homogeneous function of the direction y. Moreover, Elster and Thierfelder (1988a) prove the following properties.

Theorem 1

(1.1) If $K\left(\mathrm{epi} f,\ (x, f(x))\right)$ is convex, then $f^K(x,\cdot)$ is sublinear.

(1.2) $\mathrm{epi} f^K,\ (x,\cdot) = \left\{(y,\beta) \in \mathbb{R}^n \times \mathbb{R} : \forall \varepsilon > 0, (y, \beta + \varepsilon) \in K\left(\mathrm{epi} f,\ (x, f(x))\right)\right\}$.

(1.3) If $K\left(\mathrm{epi} f,\ (x, f(x))\right)$ is closed, it holds $\mathrm{epi} f^K(x,\cdot) = K\left(\mathrm{epi} f,\ (x, f(x))\right)$ and therefore $f^K(x,\cdot)$ is lower semicontinuous.

(1.4) $\mathrm{epi}^\circ f^K(x,\cdot) = \left\{(y,\beta) \in \mathbb{R}^n \times \mathbb{R} : \forall \varepsilon > 0, (y, \beta - \varepsilon) \in K\left(\mathrm{epi} f,\ (x, f(x))\right)\right\}$, where $\mathrm{epi}^\circ f^K = \left\{(y,\alpha) : \alpha > f^K(x,y)\right\}$.

(1.5) If $K\left(\mathrm{epi} f,\ (x, f(x))\right)$ is open, it holds $\mathrm{epi}^\circ f^K(x,\cdot) = K\left(\mathrm{epi} f,\ (x, f(x))\right)$, and $f^K(x,\cdot)$ is upper semicontinuous.

Definition 3 allows to generate many particular notions of generalized directional derivatives; in particular, by using the previously defined local cone approximations, it is possible to obtain the following types of *K-derivatives.*

Theorem 2

Let $f : \mathbb{R}^n \to \mathbb{R}$, and $x \in \mathbb{R}^n$. Then

- $$f^T(x,y) = \liminf_{t\to 0^+; y'\to y} \frac{f(x+ty') - f(x)}{t} = f_-^{DH}(x,y)$$
  is the *lower Dini-Hadamard directional derivative.*
- $$f^I(x,y) = \limsup_{t\to 0^+; y'\to y} \frac{f(x+ty') - f(x)}{t} = f_+^{DH}(x,y)$$
  is the *upper Dini-Hadamard directional derivative.*
- $$f^Z(x,y) = \limsup_{t\to 0^+} \frac{f(x+ty) - f(x)}{t} = f_+^{D}(x,y)$$
  is the *upper Dini directional derivative.*
- $$f^F(x,y) = \liminf_{t\to 0^+} \frac{f(x+ty) - f(x)}{t} = f_-^{D}(x,y)$$
  is the *lower Dini directional derivative.*

So, when $F = Z$, we have the *classical directional derivative* $f'(x,y)$.

- $$f^A(x,y) = \sup_{\delta>0} \limsup_{t\to 0^+} \inf_{y'\in N(y,\delta)} \frac{f(x+ty') - f(x)}{t}$$
  is the *lower Ursescu directional derivative* (see Ursescu (1982)).
- $$f^Q(x,y) = \inf_{\delta>0} \liminf_{t\to 0^+} \sup_{y'\in N(y,\delta)} \frac{f(x+ty') - f(x)}{t}$$
  is the *upper Ursescu directional derivative.*
- $$f^H(x,y) = \limsup_{(x',\alpha)\to_f x;\ t\to 0^+} \frac{f(x'+ty) - \alpha}{t},$$
  where the expression $(x',\alpha) \to_f x$ means that $(x',\alpha) \in \text{epi} f$, $x' \to x$ and $\alpha \to f(x)$, is the *generalized Clarke directional derivative.*
- $$f^{TC}(x,y) = \sup_{\delta>0} \limsup_{(x',\alpha)\to_f x; t\to 0^+} \inf_{y'\in N(y,\delta)} \frac{f(x'+ty') - \alpha}{t}$$
  is the *generalized Clarke-Rockafellar directional derivative*, usually denoted by $f^{\uparrow}(x,y)$.
- $$f^E(x,y) = \limsup_{t\to 0^+; y'\to y; (x',\alpha)\to_f x} \frac{f(x'+ty') - \alpha}{t}$$
  is the *epilipschitzian directional derivative.*

It must be noted that when f is lower semicontinuous $(x',\alpha) \to {}_f x$ becomes $x' \to {}_f x$, and if f is continuous, it becomes simply $x' \to x$. If f is locally Lipschitz, then

$$f^H(x,y) = f^{TC}(x,y) = \limsup_{t\to 0^+; x'\to x} \frac{f(x'+ty) - f(x')}{t},$$

i.e. we obtain the *classical Clarke directional derivative*, usually denoted by $f^o(x,y)$. Moreover, in this case

$$f^A(x,y) = f^T(x,y) = f^F(x,y)$$

and

$$f^Q(x,y) = f^I(x,y) = f^Z(x,y).$$

Together with the axiomatic notion of K-directional derivative, it is possible to introduce an axiomatic notion of *K-subdifferential* (see Elster and Thierfelder (1985, 1988a)).

Definition 4

Let $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$ and K a local cone approximation.

The set

$$\partial_K f(x) = \left\{x^* \in \mathbb{R}^n : f^K(x,y) \geq x^*y, \forall y \in \mathbb{R}^n\right\}$$

is said *K-subdifferential* of f at x and the elements $x^* \in \partial_K f(x)$ are said *K-subgradients.*

We remark that $\partial_K f(x)$ can be the empty set; moreover, it is a closed and convex set.

Theorem 3 (Elster and Thierferlder (1985, 1988a))

Let $f : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$ and K a local cone approximation. Then it holds

$$\partial_K f(x) = \{x^* \in \mathbb{R}^n : (x^*, -1) \in K^*(\text{epi} f, (x, f(x))\},$$

were K^* is the *polar cone of* K.

Moreover, it holds

$$\partial_K f(x) = \partial_{\text{cl}(\text{conv}K)} f(x).$$

Theorem 4 (Elster and Thierferlder (1985, 1988a))

If K is closed and convex, then it holds

(4.1) $f^K(x,y) \geq 0$ if and only if $\partial_K f(x) \neq \emptyset$.

(4.2) If $f^K(x,0) = 0$, then $f^K(x,y) = \sup\{x^*y : x^* \in \partial_K f(x)\}, \forall y \in \mathbb{R}^n$.

(4.3) $\exists U(0) : f^K(x,y) \leq 1, \forall y \in U(0) \Rightarrow \partial_K f(x)$ is compact.

(4.4) $\exists U(0) : \left|f^K(x,y)\right| \leq 1, \forall y \in U(0) \Rightarrow f^K(x,y) = \max\{x^*y : x^* \in \partial_K f(x)\}, \forall y \in \mathbb{R}^n$.

For other properties of the K-subdifferential and its relations with the K-derivatives, see the quoted papers of Elster and Thierfelder. We now take into consideration the problem $(\mathcal{P})$, supposing that f and every $g_i, i \in I = \{1, 2, ..., m\}$ are functions defined on $\mathbb{R}^n$ and taking values in $[-\infty, +\infty]$. We shall be concerned in particular with necessary optimality conditions and constraint qualifications, in terms of K-derivatives and K-subdifferentials. We state first some results and definitions.

The sets

$D_f^K(x) = \left\{y \in \mathbb{R}^n : f^K(x,y) < 0\right\}$ (*cone of descending directions* of f at x);

$C_f^K(x) = \left\{y \in \mathbb{R}^n : f^K(x,y) \leq 0\right\}$ (*linearizing cone* of f at x);

$D_I^K(x) = \bigcap_{i \in I} D_{g_i}^K(x)$;

$C_I^K(x) = \bigcap_{i \in I} C_{g_i}^K(x)$

are cones (as f^K is positively homogeneous) and, in particular, they are convex if K is convex.

Definition 5

The set

$$\begin{aligned} B_I^K(x) &= \left\{ x^* \in \mathbb{R}^n : x^* = \sum_{i\in I} \lambda_i x^i, \lambda_i \geq 0, x^i \in \partial_K g_i(x), i \in I \right\} = \\ &= \left\{ x^* \in \mathbb{R}^n : x^* = \sum_{i\in I} \lambda_i \partial_K g_i(x), \lambda_i \geq 0, i \in I \right\} = \\ &= \sum_{i\in I} \text{cone}\partial_K g_i(x) \end{aligned}$$

is called the *cone of K-subgradients of the functions $g_i, i \in I$*, at x.
We agree that $B_\emptyset^K = \{0\}$.

With reference to $(\mathcal{P})$, we denote by $S = \{x : g_i(x) \leq 0\}$ its *feasible set* and by $I(x^0)$ the set of the *active constraints* at the feasible point x^0:

$$I(x^0) = \left\{ i \in I : g_i(x^0) = 0 \right\}.$$

We suppose that $S \neq \emptyset$ and that $|f(x^0)| < +\infty, |g_i(x^0)| < +\infty, \forall i \in I$.

Moreover, following Elster and Thierfelder (1988a) we always assume that every g_i is upper semicontinuous for every $i \in I \backslash I(x^0)$.

Following Elster and Thierfelder (1985, 1988a) and Giorgi, Guerraggio and Thierfelder (2004) we now give, for the problem $(\mathcal{P})$, necessary optimality conditions of the Kuhn-Tucker-type, in terms of K-derivatives and K-subdifferentials.

We suppose that the cone approximations have the following additional properties:

(A_1) $K(\cdot,\cdot)$ is convex and closed;

(A_2) $x \in \text{cl}(M) \Leftrightarrow 0 \in K(M,x)$;

(A_3) $K(\cdot,\cdot) \subset T(\cdot,\cdot)$;

(A_4) $\text{int}(K(M,x)) \subset I(M,x)$.

The following general necessary optimality conditions hold.

Theorem 5

If $x^0 \in S$ is a local solution of $(\mathcal{P})$, then it holds

(5.1) $D_f^{\text{int}(K)}(x^0) \cap K(S,x^0) = \emptyset$;

(5.2) $D_f^K(x^0) \cap \text{int}(K(S,x^0)) = \emptyset$.

Theorem 6

If $x^0 \in S$ is a local solution of $(\mathcal{P})$ and if one of the following two conditions

(B_1) $\text{dom} f^{\text{int}(K)}(x^0,\cdot) \cap K(S,x^0) \neq \emptyset$;

(B_2) $\text{dom} f^K(x^0,\cdot) \cap \text{int}(K(S,x^0)) \neq \emptyset$,

is satisfied, then

$$0 \in \partial_K f(x^0) + K^*(S,x^0).$$

From this last theorem it is possible to derive easily the following Kuhn-Tucker-type necessary optimality conditions for $(\mathcal{P})$ in terms of axiomatic K-derivatives and

K-subdifferentials. Moreover, it is possible to introduce some constraint qualifications, as well expressed in terms of K-derivatives and K-subgradients.

Theorem 7

Let $x^0 \in S$ be a local solution of $(\mathcal{P})$ and let the following constraint qualification be satisfied

$(\mathbf{CQ})_1 \quad K^*(S, x^0) \subset B^K_{I(x^0)}(x^0)$ and either (B_1) or (B_2).

Then, there exist multipliers $\lambda_i \geq 0$, $i \in I(x^0)$, such that

(7.1) $\quad 0 \in \partial_K f(x^0) + \sum_{i \in I(x^0)} \lambda_i \partial_K g_i(x^0);$

(7.2) $\quad f^K(x^0, y) + \sum_{i \in I(x^0)} \lambda_i g_i^K(x^0, y) \geq 0, \forall y \in \mathbb{R}^n.$

Now we shall give some other constraint qualifications which are sufficient for $(\mathbf{CQ})_1$. We denote by $(\mathcal{R})$ the following "regularity" condition

$$(\mathcal{R}) \qquad \begin{cases} \text{Either } (B_1) \text{ or } (B_2) \text{ is satisfied;} \\ \partial_K g_i(x^0) \neq \emptyset, \forall i \in I(x^0); \\ B^K_{I(x^0)}(x^0) \text{ is closed.} \end{cases}$$

We formulate the following constraint qualifications.

$(\mathbf{CQ})_2$ (Guignard-Gould-Tolle-type c.q.)

$$K^*(S, x^0) \subset \left(C^K_{I(x^0)}(x^0)\right)^*, \quad (\mathcal{R}) \text{ is verified.}$$

$(\mathbf{CQ})_3$ (Abadie-type c.q.)

$$C^K_{I(x^0)}(x^0) \subset K(S, x^0), \quad (\mathcal{R}) \text{ is verified.}$$

$(\mathbf{CQ})_4$ (Slater-type c.q.)

$$D^{\mathrm{int}(K)}_{I(x^0)}(x^0) \neq \emptyset, \quad (\mathcal{R}) \text{ is verified.}$$

$(\mathbf{CQ})_5$ (Slater-type c.q.)

$$\mathrm{dom} f^K(x^0, \cdot) \cap D^{\mathrm{int}(K)}_{I(x^0)}(x^0) \neq \emptyset, \quad \partial_K g_i(x^0) \neq \emptyset, \forall i \in I(x^0), B^K_{I(x^0)}(x^0) \text{ closed.}$$

By supposing that, besides (A_1)...(A_4), also the following condition

$(A_5) \quad D^K_{I(x^0)}(x^0) \subset K(S, x^0)$

is satisfied, then it is possible to prove the following implications and coimplications (Elster and Thierfelder (1988a), Giorgi, Guerraggio and Thierfelder (2004))

$$(\mathbf{CQ})_5 \Rightarrow (\mathbf{CQ})_4 \Rightarrow (\mathbf{CQ})_3 \Leftrightarrow (\mathbf{CQ})_2 \Rightarrow (\mathbf{CQ})_1.$$

Elster and Thierfelder (1985, 1988a) give also Fritz John-type conditions for $(\mathcal{P})$. From these conditions, under appropriate constraint qualifications, it is then possible to obtain the same Kuhn-Tucker-type conditions of Theorem 7.

Theorem 8

Let $x^0 \in S$ be a local solution of $(\mathcal{P})$ and let the assumptions $(A_1)...(A_5)$ be verified. Then

(8.1) there exist multipliers $\lambda_i \geq 0, i \in \{0\} \cup I(x^0)$, not all zero, such that

$$\lambda_0 f^{\mathrm{int}(K)}(x^0, y) + \sum_{i \in I(x^0)} \lambda_i g_i^K(x^0, y) \geq 0,$$
$$\forall y \in \mathrm{dom} f^{\mathrm{int}(K)}(x^0, \cdot) \cap (\bigcap_{i \in I(x^0)} \mathrm{dom} g_i^K(x^0, \cdot));$$

(8.2) there exist multipliers $\lambda_i \geq 0, i \in \{0\} \cup I(x^0)$, not all zero, such that

$$\lambda_0 f^K(x^0, y) + \sum_{i \in I(x^0)} \lambda_i g_i^{\mathrm{int}(K)}(x^0, y) \geq 0,$$
$$\forall y \in \mathrm{dom} f^K(x^0, \cdot) \cap (\bigcap_{i \in I(x^0)} \mathrm{dom} g_i^{\mathrm{int}(K)}(x^0, \cdot)).$$

By means of the following conditions (in Elster and Thierfelder (1988a) there is a misprint):

(B_3) either $\exists i_0 \in I(x^0)$ such that

$$\mathrm{dom} g_{i_0}^K(x^0, y) \bigcap_{i \in I(x^0) \setminus \{i_0\}} \mathrm{dom} g_i^{\mathrm{int}(K)}(x^0, y) \cap \mathrm{dom} f^{\mathrm{int}(K)}(x^0, y) = \mathbb{R}^n,$$

(B_3') or $\mathrm{dom} f^K(x^0, y) \bigcap_{i \in I(x^0)} \mathrm{dom} g_i^{\mathrm{int}(K)}(x^0, y) = \mathbb{R}^n$

we can give a sharpened version of Theorem 8.

Theorem 9

Let $x^0 \in S$ be a local solution of $(\mathcal{P})$ and let either (B_3) or (B_3') be satisfied, besides $(A_1)...(A_5)$. Then there exist multipliers $\lambda_i \geq 0, i \in \{0\} \cup I(x^0)$, not all zero, such that

(9.1) $0 \in \lambda_0 \partial_K f(x^0) + \sum_{i \in I(x^0)} \lambda_i \partial_K g_i(x^0)$.

(9.2) $\lambda_0 f^K(x^0, y) + \sum_{i \in I(x^0)} \lambda_i g_i^K(x^0, y) \geq 0, \forall y \in \mathbb{R}^n$.

If the following constraint qualification

$(\mathbf{CQ})_6$ $(D^K_{I(x^0)}(x^0) \neq \emptyset$ and it holds either (B_3) or $(B_3'))$

is satisfied, then we have, from Theorem 9, the following result, of the Kuhn-Tucker-type.

Theorem 10

Let $x^0 \in S$ be a local solution of $(\mathcal{P})$, let the assumptions of Theorem 8 be verified and let $(\mathrm{CQ})_6$ be satisfied. Then $\lambda_0 \neq 0$ (i.e. $\lambda_0 > 0$, i.e. $\lambda_0 = 1$) in (9.1) and (9.2) of Theorem 9.

A relationship between $(\mathrm{CQ})_6$ and conditions $(\mathrm{CQ})_1 \ldots (\mathrm{CQ})_5$ cannot be established. Indeed, the condition $(\mathrm{CQ})_6$ is rather strong and it is not necessary for the validity of $(\mathrm{CQ})_1 \ldots (\mathrm{CQ})_5$.

If we use the conditions

(B_4) $\quad \mathrm{dom} f_i^{\mathrm{int}(K)}(x^0, \cdot) \bigcap_{i \in I(x^0)} \mathrm{dom} g_i^{\mathrm{int}(K)}(x^0, \cdot) = \mathbb{R}^n$

and

$(\mathbf{CQ})_7$ $\quad (B_4) \cap (D^K_{I(x^0)}(x^0) \neq \emptyset$

it is possible to show that

$$(\mathrm{CQ})_7 \Rightarrow (\mathrm{CQ})_6;$$

$$(\mathrm{CQ})_7 \Rightarrow (\mathrm{CQ})_5.$$

Remark 2

The constraint qualifications examined and the related optimality conditions are true generalizations of well-known results established by various authors for nonsmooth optimization problems of the type of $(\mathcal{P})$. See the conclusive sentences of the paper of Elster and Thierfelder (1985).

3. G-semidifferentiable functions

Another axiomatic approach to the construction of "a sort of container of several existing concepts [of differentiability] and suitable for achieving necessary optimality conditions" (Giannessi (2005)) is developed by Giannessi (1989). See also Giannessi (2005), Pappalardo (1992, 1993a, 1993b),Yen (1995), Pappalardo and Uderzo (1999). In the present Section we follow mainly Rocca (1994). As in the previous Section, also here we give the notions in $\mathbb{R}^n$, pointing out that the original treatment of Giannessi is performed in Hilbert spaces.

Let $X \subset \mathbb{R}^n$, $x^0 \in X$, $y = x - x^0$ $(x \in X)$, $D = \mathrm{cone}(X - x^0)$ or at least D is given by the intersection of the closed unit ball with $\mathrm{cone}(X - x^0)$, in order that the next definitions make sense.

We denote by $\mathcal{G}$ the class of functions $p : X \times D \to \mathbb{R}$ linearly homogenous (i.e. positively homogeneous of degree one) with respect to the second argument, and with G a subset of $\mathcal{G}$.

Definition 6

A function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is said *upper G-semidifferentiable* at x^0 if there exist two functions $p \in G \subset \mathcal{G}$ and $\varepsilon : X \times D \to \mathbb{R}$ such that

(i) $\quad \limsup_{y \to 0} \varepsilon(x^0, y) / \|y\| \leq 0;$

(ii) $\quad f(x) = f(x^0) + p(x^0, y) + \varepsilon(x^0, y)$, $\forall y \in D$;

(iii) $\quad$ for each $(\overline{p}, \overline{\varepsilon})$ which satisfies conditions (i) and (ii), with $\overline{p} \in G$, we have

$$\mathrm{epi}\, \overline{p} \subset \mathrm{epi}\, p.$$

The function $p(x^0, y/\|y\|)$ is called the *upper directional G-semiderivative* of f at x^0 in the direction y.

Definition 7

A function $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is said *lower G-semidifferentiable* at x^0 when there exist two functions $p \in G \subset \mathcal{G}$ and $\varepsilon : X \times D \to \mathbb{R}$ such that

(iv) $\liminf_{y \to 0} \varepsilon(x^0, y)/\|y\| \geq 0$;

(v) $f(x) = f(x^0) + p(x^0, y) + \varepsilon(x^0, y)$, $\forall y \in D$;

(vi) for each $(\overline{p}, \overline{\varepsilon})$ which satisfies conditions (iv) and (v), with $\overline{p} \in G$, we have

$$\text{hypo}\, \overline{p} \subset \text{hypo}\, p.$$

The function $p(x^0, y/\|y\|)$ is called the *lower directional G-semiderivative* of f at x^0 in the direction y.

Definition 8

A function $f : X \to \mathbb{R}$ which is at the same time upper G-semidifferentiable and lower G-semidifferentiable at x^0, with the same function p, so that it holds

(vii) $\lim_{y \to 0} \varepsilon(x^0, y)/\|y\| = 0$,

is said to be *G-differentiable* at x^0.

This class of functions was introduced independently by Giannessi (1989) and by Robinson (1987) (this last author calls them "B-differentiable functions", where "B" should stand for "Bouligand").

If $G = \mathcal{L} \subset \mathcal{G}$, where $\mathcal{L}$ is the subset of $\mathcal{G}$ given by *linear functions*, and the property (vii) holds, then f turns out to be differentiable at x^0 in the usual Fréchet sense (in this case D is required to be a linear subspace). A particular attention must be given to the class of $\mathcal{C}-$semidifferentiable functions, where $\mathcal{C}$ is the subset of $\mathcal{G}$ given by *convex functions*, with respect to the second variable (in this case D is required to be a convex cone).

Definition 9

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$, $G \subset \mathcal{C}$, and let f be upper (resp. lower) G-semidifferentiable at x^0; let p be its upper (resp. lower) directional G-semiderivative.

We call *G-generalized subdifferential* of f at x^0, denoted $\partial_G f(x^0)$, the subdifferential, at $y = 0$, of the convex function $p(x^0, y)$:

$$\partial_G f(x^0) = \partial p(x^0, 0).$$

The previous definition makes sense even if G does not belong to $\mathcal{C}$, but with a convex G-semiderivative of f at x^0. Also in this case we can speak of G-generalized subdifferential.

An element $v \in \partial_G f(x^0)$ is said *G-generalized subgradient* of f at x^0.

Theorem 11 (Giannessi (1989))

Let $X \subset \mathbb{R}^n$ be open and convex.

(11.1) If $f : X \to \mathbb{R}$ is an upper $\mathcal{C}$-semidifferentiable function, then

$$\partial_{\mathcal{C}}(\alpha f)(x) = \alpha \partial_{\mathcal{C}} f(x),\ \forall \alpha > 0,\ \forall x \in X,$$

(the same equality holds for $\alpha = 0$, whenever $\partial_{\mathcal{C}} f(x) \neq \emptyset$).

(11.2) If $f_i : X \to \mathbb{R}$, $i = 1, 2$, are upper $\mathcal{C}$-semidifferentiable functions, then

$$\partial_{\mathcal{C}} \left[f_1(x) + f_2(x)\right] \subset \partial_{\mathcal{C}} f_1(x) + \partial_{\mathcal{C}} f_2(x);$$

(11.3) $\partial_{\mathcal{C}} f(x)$ is a convex set.

(11.4) If f is convex, then $\partial_{\mathcal{C}} f(x) = \partial f(x)$;
moreover, f is $\mathcal{C}$-differentiable at any $x^0 \in X$ and its unique directional $\mathcal{C}$-derivative coincides with the directional derivative of f.

The next result shows that the axiomatic approach of Giannessi includes also the approach of Clarke for locally Lipschitz functions.

Theorem 12

Let us consider the following subset of $\mathcal{G}$:

$$\mathcal{C}^0 = \left\{p \in \mathcal{G} :\ \text{epi}p \subset \text{cl}H(\text{epi}f, (x^0, f(x^0))\right\},$$

where $H(M, x^0)$ is the hypertangent cone at $x^0 \in \text{cl}\,(M)$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz at x^0, then f is upper $\mathcal{C}^0$-semidifferentiable at x^0, with unique upper directional $\mathcal{C}^0$-semiderivative given by the generalized directional Clarke derivative $f^o(x^0, y)$.

For other insights and connections between G-semidifferentiable functions (in Euclidean spaces) and Hadamard, Dini-Hadamard and Clarke directional derivatives, see Yen (1995) and Pappalardo and Uderzo (1999).

In this section we consider necessary optimality conditions of the Kuhn-Tucker-type for $(\mathcal{P})$, where the functions involved are assumed to be G-semidifferentiable. We first consider a programming problem with a set constraint:

$(\mathcal{P}_1)$ $\quad \min f(x), \quad x \in C \subset X \subset \mathbb{R}^n,$

where $f : X \subset \mathbb{R}^n \to \mathbb{R}$ and C is any subset of X (C not necessarily open).

Theorem 13

Let $x^0 \in C$ be a local solution of $(\mathcal{P}_1)$ and let f be an upper G-semidifferentiable function at x^0. If the upper directional G-semiderivative of f at x^0, $p(x^0, \cdot)$ is upper semicontinuous , it holds

$$p(x^0, y) \geq 0,\ \forall y \in T(C; x^0) \cap D.$$

Proof

If $y = 0$, from the homogeneity of $p(x^0, y)$ it follows at once $p(x^0, 0) = 0$.
Let us now suppose that there exists also $y \neq 0$, with $y \in T(C; x^0) \cap D$. By the definition of upper directional G-semiderivative, we can write

$$f(x) = f(x^0) + p(x^0, x - x^0) + \varepsilon(x^0, x - x^0),\ \forall x \in X,$$

with $\limsup_{x \to x^0} \dfrac{\varepsilon(x^0, x - x^0)}{\|x - x^0\|} \leq 0.$

Let us now consider the sequences $\{x^k\} \to x^0$, $x^k \in C \subset X$, and $\{\lambda_k\}$, $\lambda_k \in \mathbb{R}_+$, such that $\{\lambda_k(x^k - x^0)\} \to y$.

Being $y \neq 0$, there exists an integer N such that $x^k \neq x^0$, $\forall k \geq N$; without loss of generality, we can then suppose $x^k \neq x^0$ for any k. We have therefore, for any k:

$$f(x^k) - f(x^0) = p(x^0, x^k - x^0) + \varepsilon(x^0, x^k - x^0),$$

from which, taking the linear homogeneity of $p(x^0, \cdot)$ into account,

$$\lambda_k \left(f\left(x^k\right) - f\left(x^0\right)\right) = p\left(x^0, \lambda_k\left(x^k - x^0\right)\right) + \left\|\lambda_k(x^k - x^0)\right\| \cdot \gamma(x^k - x^0), \tag{1}$$

where

$$\gamma(x^k - x^0) = \frac{\varepsilon(x^0, x - x^0)}{\|x - x^0\|}.$$

As the first member of (1) is non negative, also the second member will be non negative. Moreover, denoting

$$\bar{l} = \limsup_{k \to +\infty} \gamma(x^k - x^0),$$

we can find a subsequence $\{x^{k_n}\}$ of $\{x^k\}$ such that

$$\limsup_{n \to +\infty} \frac{\varepsilon(x^0, x^{k_n} - x^0)}{\|x^{k_n} - x^0\|} = \bar{l} \leq 0.$$

The sequence $\{\lambda_{k_n}(x^{k_n} - x^0)\}$ converges to y. Therefore, from the inequality

$$p(x^0, \lambda_{k_n}\left(x^{k_n} - x^0\right)) + \left\|\lambda_{k_n}(x^{k_n} - x^0)\right\| \cdot \gamma(x^{k_n} - x^0) \geq 0,$$

taking the limit for $n \to +\infty$ and taking the upper semicontinuity of $p(x^0, \cdot)$ into account, we get

$$p(x^0, y) + \|y\| \cdot \bar{l} \geq 0,$$

from which we obtain the thesis. □

Remark 3

- Being $T(C, x^0) \subset \text{cl}(\text{cone}(C - x^0) \subset \text{cl}(D)$, it follows that, if D is closed, the thesis of the theorem can be replaced by the relation

$$p(x^0, y) \geq 0, \ \forall y \in T(C, x^0).$$

- Theorem 13 holds, more generally, for any upper semicontinuous function $p \in G$, which verifies relations (i) and (ii) of Definition 6.

- If $f : X \subset \mathbb{R}^n \to \mathbb{R}$, X open, is an $\mathcal{L}$-differentiable function, the assumptions of Theorem 13 are satisfied. In this case we deduce the necessary optimality condition of Guignard-Gould-Tolle (Guignard (1969), Gould and Tolle (1971)):

$$-\nabla f(x^0) \in T^*(C, x^0),$$

where $T^*(C, x^0)$ is the (negative) polar cone of $T(C, x^0)$.

- If C is an open set or also if $x^0 \in \text{int}(C)$, we have $T(C, x^0) = \mathbb{R}^n = D$. In this case we obtain a generalized necessary optimality condition for an unconstrained minimization problem:

$$p(x^0, y) \geq 0, \ \forall y \in \mathbb{R}^n.$$

Theorem 14

Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$, an upper G-semidifferentiable function at $x^0 \in C$ and let Γ be a convex subcone of $T(C, x^0)$, such that $\text{relint}(\Gamma) \cap \text{relint}(D) \neq \emptyset$. If the upper directional G-semiderivative of f at x^0, $p(x^0, \cdot)$, is convex and upper semicontinuous, then, if $x^0 \in S$ is a local solution of $(\mathcal{P}_1)$, it holds

$$0 \in \partial_G f(x^0) + \Gamma^* + D^*. \tag{2}$$

Proof

We first note that, by well-known results of the Convex Analysis, from the assumption $\text{relint}(\Gamma) \cap \text{relint}(D) \neq \emptyset$, we get

$$\begin{aligned}\text{relint}(D) \cap \text{relint}(\text{cl}\,(\Gamma) \cap D) &= \text{relint}(D) \cap \{\text{relint}(\text{cl}\,(\Gamma)) \cap \text{relint}(D)\} = \\ &= \text{relint}(D) \cap \text{relint}(\text{cl}\,(\Gamma) = \text{relint}(D) \cap \text{relint}(\Gamma) \neq \emptyset.\end{aligned}$$

As, by Theorem 13, it results

$$p(x^0, y) \geq 0, \ \forall y \in T(C, x^0) \cap D,$$

being $\text{cl}\,(\Gamma) \subset T(C, x^0)$, it must also result

$$p(x^0, y) \geq 0, \ \forall y \in \text{cl}\,(\Gamma) \cap D.$$

As $0 \in \text{cl}\,(\Gamma) \cap D$, we deduce that $y^* = 0$ is a minimum point for $p(x^0, \cdot)$ on $\text{cl}\,(\Gamma) \cap D$. It is well-known that, if h is a convex function and $B \subset \mathbb{R}^n$ is a convex set, if $\text{relint}(\text{dom}(h)) \cap \text{relint}(B) \neq \emptyset$ ($\text{dom}(h)$ being the domain of h), then a necessary and sufficient condition for $\overline{x} \in B$ to be a minimum point for h on B is

$$0 \in \partial h(\overline{x}) + N(B, \overline{x}),$$

where $N(B, \overline{x})$ is the usual *normal cone* at $\overline{x}$ with respect to B (see Rockafellar (1970)).

If $\overline{x} = 0$, $N(B, 0)$ coincides with the (negative) polar cone of B. Then, a necessary and sufficient condition for $y^* = 0$ to be a minimum point for $p(x^0, \cdot)$ on $\mathrm{cl}\,(\Gamma) \cap D$ is

$$0 \in \partial p(x^0, 0) + (\mathrm{cl}\,(\Gamma) \cap D)^*,$$

that is, taking the assumption $\mathrm{relint}(\Gamma) \cap \mathrm{relint}(D) \neq \emptyset$ into account,

$$0 \in \partial p(x^0, 0) + (\mathrm{cl}\,(\Gamma))^* + D^* = \partial p(x^0, 0) + \Gamma^* + D^*.$$

For the definition of $\partial_G f(x^0)$, this is equivalent to require

$$0 \in \partial_G f(x^0) + \Gamma^* + D^*.$$

□

Remark 4

- If D is closed, from $\Gamma \subset T(C, x^0) \subset D$, under the assumptions of Theorem 14 it follows that relation (2) becomes

$$0 \in \partial_G f(x^0) + \Gamma^*.$$

- The result of Theorem 14 becomes sharper if $T(C, x^0)$ is a convex cone, as in this case $\Gamma = T(C, x^0)$. A sufficient condition to have $T(C, x^0)$ convex is that C is convex or also that C is *locally convex* at x^0 (i.e. there exists a neighborhood of x^0, $B(x^0)$, such that the set $C \cap B(x^0)$ is convex). If $T(C, x^0)$ is not a convex cone, it is however always possible to find convex subcones of $T(C, x^0)$: e.g. the *Clarke tangent cone* (Clarke (1983)) or the *Michel-Penot prototangent cone*, larger than the Clarke tangent cone (Michel and Penot (1984)).

- If $f : X \subset \mathbb{R}^n \to \mathbb{R}$, X open and convex, is a convex function, Theorem 14 collapses to the well-known condition

$$0 \in \partial f(x^0) + N(C, x^0).$$

  Similarly, if $f : X \to \mathbb{R}$, X open, is a locally Lipschitzian function at $x^0 \in X$, then Theorem 14 collapses to the Clarke necessary optimality condition

$$0 \in \Omega f(x^0) + (TC(C, x^0))^*,$$

  where $\Omega = \{v \in \mathbb{R}^n : f^o(x^0, y) \geq v \cdot y,\ \forall y \in \mathbb{R}^n\}$ is the *Clarke generalized gradient* at x^0.

Definition 10

Let $g_1, g_2, ..., g_m$ be upper G-semidifferentiable at x^0, with upper directional G-semiderivatives $p_1, p_2, ..., p_m$ convex. Let $\partial_G g_i(x^0) \neq \emptyset$ for every $i \in I = \{1, 2, \ldots, m\}$. The *cone of the G-generalized subdifferential* of $g_1, g_2, ..., g_m$ at x^0 is the set

$$M(x^0, I) = \sum_{i=1}^{m} \text{cone}\left(\partial_G g_i(x^0)\right).$$

Theorem 15

Let $x^0 \in S$ be a local solution of $(\mathcal{P})$ and let f be upper G-semidifferentiable at x^0 with a convex and continuous upper directional G-semiderivative. Let every g_i, $i \in I(x^0)$, be upper G-semidifferentiable at x^0 with a convex upper directional G-semiderivative $p_i(x^0, \cdot)$. If $T(S, x^0)$ is convex, $\text{relint}(D) \cap \text{relint}(T(S, x^0)) \neq 0$, $\partial_G g_i(x^0) \neq \emptyset$, $\forall i \in I(x^0)$, and the following constraint qualification holds

$(\mathbf{CQ})_{\text{I}}$ $\quad M(x^0, I(x^0)) \supset T^*(S, x^0) + D^*$,

then there exist multipliers $\lambda_i \geq 0$, $i \in I(x^0)$, such that

$$0 \in \partial_G f(x^0) + \sum_{i \in I(x^0)} \lambda_i \partial_G g_i(x^0).$$

Proof

From the previous theorem it follows that

$$0 \in \partial_G f(x^0) + T^*(S, x^0) + D^*,$$

that is, for the assumed constraint qualification,

$$0 \in \partial_G f(x^0) + M(x^0, I(x^0)).$$

Therefore the thesis follows at once. □

We now take into consideration other constraint qualifications and continue to assume that every g_i, $i \in I(x^0)$, is an upper G-semidifferentiable function at x^0, with a convex upper directional G-semiderivative, that $T(S, x^0)$ is convex and that $\partial_G g_i(x^0) \neq \emptyset$, $\forall i \in I(x^0)$.

We consider the following constraint qualifications $(\text{CQ})_{\text{II}}$...$(\text{CQ})_{\text{VIII}}$.

$(\mathbf{CQ})_{\text{II}}$ $\quad (A')^* \supset T^*(S, x^0)$, $M(x^0, I(x^0))$ is closed,
where $A' = \bigcap_{i \in I(x^0)} A'_i$, $A'_i = \{y \in D : p_i(x^0, y) \leq 0\}$.
$(\text{CQ})_{\text{II}}$ may be regarded as a generalized Guignard-Gould-Tolle constraint qualification.

$(\mathbf{CQ})_{\text{III}}$ $\quad A' \subset T(S, x^0)$, $M(x^0, I(x^0))$ is closed.
$(\text{CQ})_{\text{III}}$ may be regarded as a generalized Abadie constraint qualification.

(CQ)$_{\rm IV}$ $A' \subset \mathrm{cl}(Z(S,x^0))$, $M(x^0, I(x^0))$ is closed.

We recall (see Section 1, Definition 2) that the set

$Z(S,x^0) = \{y \in \mathbb{R}^n : \exists \lambda > 0 \text{ such that } \forall t \in (0,\lambda), x^0 + ty \in S\}$

is the *cone of the feasible directions* at x^0 for S.

(CQ)$_{\rm IV}$ may be regarded as a generalized Zangwill constraint qualification.

(CQ)$_{\rm V}$ $A' \subset \mathrm{cl}(A)$, $M(x^0, I(x^0))$ is closed,

where $A = \underset{i \in I(x^0)}{\cap} A_i$, $A_i = \{y \in D : p_i(x^0, y) < 0\}$.

(CQ)$_{\rm V}$ may be regarded as a generalized Cottle constraint qualification.

(CQ)$_{\rm VI}$ $\exists \overline{y} \in D : p_i(x^0, \overline{y}) < 0, \forall i \in I(x^0)$, $M(x^0, I(x^0))$ is closed.

(CQ)$_{\rm VI}$ may be regarded as a generalized Slater constraint qualification.

(CQ)$_{\rm VII}$ $0 \notin \partial p(x^0, 0)$, $M(x^0, I(x^0))$ is closed,

where $p(x^0, y) = \max\{p_i(x^0, y), i \in I(x^0), y \in D\}$.

(CQ)$_{\rm VIII}$ There do not exist multipliers $\lambda_i \geq 0, i \in I(x^0)$, not all zero, such that

$0 \in \displaystyle\sum_{i \in I(x^0)} \lambda_i \partial_G g_i(x^0)$ and $M(x^0, I(x^0))$ is closed.

(CQ)$_{\rm VIII}$ may be regarded as a condition of generalized positive linear independence.

In order to prove the implications between the various constraint qualifications introduced, we shall make use of the following assumptions.

(a_1) $D = \mathbb{R}^n$.

(a_2) $\forall y \in D, \exists \lambda > 0 : \forall t \in (0, \lambda), x^0 + ty \in X$, where X is the common domain of f and every g_i, $i = 1, 2, ..., m$.

(a_3) The upper directional G-semiderivatives of the functions $g_i, i \in I(x^0)$, are continuous.

(a_4) $\mathrm{relint}(T(S, x^0)) \cap \mathrm{relint}(D) \neq \emptyset$.

The notation " $\overset{a_i}{\Longrightarrow}$ " means that the related implication holds under the assumption (a_i).

Theorem 16

The following relations hold.

$$(\mathrm{CQ})_{\rm VIII} \;\underset{\Longleftarrow}{\overset{a_1}{\Longrightarrow}}\; (\mathrm{CQ})_{\rm VII} \;\underset{\Longleftarrow}{\Longrightarrow}\; (\mathrm{CQ})_{\rm VI} \Longrightarrow (\mathrm{CQ})_{\rm V} \overset{a_2}{\Longrightarrow} (\mathrm{CQ})_{\rm IV} \Longrightarrow$$

$$\Longrightarrow (\mathrm{CQ})_{\rm III} \;\underset{\overset{a_3}{\Longleftarrow}}{\Longrightarrow}\; (\mathrm{CQ})_{\rm II} \overset{a_1, a_2, a_3}{\Longrightarrow} (\mathrm{CQ})_{\rm I} .$$

Proof

$\circ \qquad (\mathrm{CQ})_{\mathrm{VIII}} \underset{\Longleftarrow}{\overset{a_1}{\Longrightarrow}} (\mathrm{CQ})_{\mathrm{VII}}$

It easily proved that $\mathrm{conv}\{\partial_G g_i(x^0), i \in I(x^0)\} \subset \partial p(x^0, 0)$. From this inclusion we have that, if $0 \notin \partial p(x^0, 0)$, also $0 \notin \mathrm{conv}\{\partial g_i(x^0), i \in I(x^0)\}$. If $D = \mathbb{R}^n$, it follows that $\partial_G g_i(x^0) = \partial p_i(x^0, 0)$ is a convex and compact set, for every $i \in I(x^0)$. As it is well known that

$$\partial p(x^0, 0) = \mathrm{conv}\left\{\partial p_i(x^0, 0), i \in I(x^0)\right\}$$

we have the implication $(\mathrm{CQ})_{\mathrm{VIII}} \Longrightarrow (\mathrm{CQ})_{\mathrm{VII}}$, under assumption a_1.

$\circ \qquad (\mathrm{CQ})_{\mathrm{VII}} \underset{\Longleftarrow}{\Longrightarrow} (\mathrm{CQ})_{\mathrm{VI}}$

Being the functions $p_i(x^0, \cdot)$ convex, also $p_i(x^0, 0)$ is convex. Then, $0 \notin \partial p(x^0, 0)$ if and only if $y = 0$ is not a minimum point for $p_i(x^0, \cdot)$, i.e. if and only if there exists $\overline{y} \in D$ such that $p(x^0, \overline{y}) < 0$, which implies $p_i(x^0, \overline{y}) < 0, \forall i \in I(x^0)$.

$\circ \qquad (\mathrm{CQ})_{\mathrm{VI}} \Longrightarrow (\mathrm{CQ})_{\mathrm{V}}$

We have to prove that $A' \subset \mathrm{cl}(A)$. The inclusion is obvious if $y \in A'$ is such that $p_i(x^0, y) < 0, \forall i \in I(x^0)$. If $y \in A'$ is such that $p_i(x^0, y) = 0$ for at least an index $i \in I(x^0)$, as there exists a vector $\overline{y} \in D$ such that $p_i(x^0, \overline{y}) < 0, \forall i \in I(x^0)$, from the convexity of $p_i(x^0, \overline{y})$, it follows

$$p_i(x^0, t\overline{y} + (1-t)y) \leq t p_i(x^0, \overline{y}) + (1-t) p_i(x^0, y) < 0, \forall t \in (0, 1], \forall i \in I(x^0).$$

Therefore, in each neighborhood of y there exist points where $p_i(x^0, \cdot) < 0, \forall i \in I(x^0)$. This shows that $y \in \mathrm{cl}(A)$.

$\circ \qquad (\mathrm{CQ})_{\mathrm{V}} \overset{a2}{\Longrightarrow} (\mathrm{CQ})_{\mathrm{IV}}$

Being $S = \bigcap_{i=1}^{m} S_i$, where $S_i = \{x \in X : g_i(x) \leq 0\}$, and being $g_i^+(x^0, y) \leq p_i(x^0, y)$, $\forall y \in D$, where $g_i^+(x^0, y)$ is the Dini upper directional derivative of g_i, at x^0 in the direction y, we get, for each $i \in I(x^0)$:

$$A_i \subset \left\{y \in D : g_i^+(x^0, y) < 0\right\} \subset Z(S_i, x^0) \cap D = Z(S_i, x^0),$$

where the last equality follows from the fact that

$$Z(S_i, x^0) \subset \mathrm{cone}(S_i - x^0) \subset D.$$

Let now be $i \notin I(x^0)$, i.e. $g_i(x^0) < 0$. Again from $g_i^+(x^0, y) \leq p_i(x^0, y), \forall y \in D$, it follows $g_i^+(x^0, y) < +\infty, \forall y \in D$. If $\limsup_{t \to 0^+} \dfrac{g_i(x^0 + ty) - g_i(x^0)}{t}$ is finite, then necessarily it will be $\limsup_{t \to 0^+} g_i(x^0 + ty) = g_i(x^0)$. Therefore, it will exist a scalar $\lambda > 0$ such that $g_i(x^0+ty) < 0, \forall t \in (0,\lambda)$. The same conclusion obviously holds if we have $\limsup_{t\to 0^+} \frac{g_i(x^0+ty)-g_i(x^0)}{t} = -\infty$. Therefore, for $g_i(x^0) < 0$, i.e. $i \notin I(x^0)$, it holds $D = Z(S_i, x^0)$. We have then,

$$A = \bigcap_{i\in I(x^0)} A_i \subset \bigcap_{i\in I(x^0)} Z(S_i,x^0) = \bigcap_{i=1}^{m} Z(S_i,x^0) = Z\left(\bigcap_{i=1}^{m} S_i, x^0\right) = Z(S,x^0),$$

from which we get $A' \subset \mathrm{cl}(A) \subset \mathrm{cl}(Z(S,x^0))$.

○ $(\mathrm{CQ})_{\mathrm{IV}} \Longrightarrow (\mathrm{CQ})_{\mathrm{III}}$

From $\mathrm{cl}(Z(S,x^0)) \subset T(S,x^0)$, the thesis follows at once.

○ $(\mathrm{CQ})_{\mathrm{III}} \underset{a_3}{\overset{\Longrightarrow}{\Longleftarrow}} (\mathrm{CQ})_{\mathrm{II}}$

From $A' \subset T(S,x^0)$ it follows $(A')^* \supset T^*(S,x^0)$. If $(\mathrm{CQ})_{\mathrm{II}}$ holds, thanks to the assumption (a_3), A' is a closed (and convex) cone.
Therefore we have $(A')^{**} = T^{**}(S,x^0) = T(S,x^0)$.

○ $(\mathrm{CQ})_{\mathrm{II}} \overset{a_1,a_2,a_3}{\Longrightarrow} (\mathrm{CQ})_{\mathrm{I}}$

As the sets A'_i have a nonempty intersection (the zero element belongs to each of them), we have, for properties of polar cones,

$$(A')^* = \mathrm{cl}\left(\sum_{i\in I(x^0)} (A_i)^*\right) = \mathrm{cl}\left(\sum_{i\in I(x^0)} \mathrm{cl}\left(\mathrm{cone}(\partial_G g_i(x^0))\right)\right) = \mathrm{cl}\left(\sum_{i\in I(x^0)} \mathrm{cone}(\partial_G g_i(x^0))\right) =$$
$$= \mathrm{cl}\left(M(x^0, I(x^0))\right) = M(x^0, I(x^0)).$$

Therefore, it holds

$$(A')^* = M(x^0, I(x^0)) \supset T^*(S,x^0) = \left(T(S,x^0)\cap \mathrm{cl}(D)\right)^* =$$
$$= T^*(S,x^0) + \left(\mathrm{cl}(D)\right)^* = T^*(S,x^0) + D^*.$$

□

4. Comparisons between the two axiomatic approaches and some remarks

The two axiomatic approaches to generalized differentiability, considered in the previous sections point out the problem of analyzing the relations between them. This has been done by Pappalardo (1992).

We follow his approach and, in particular, the following questions will be studied.

(*a*) Given a local cone approximation K, is it possible to define a subset G of $\mathcal{G}$ such that a function f is K-differentiable if and only if f is G-differentiable with f^K as G-derivative of f?

(*b*) Given a subset G of $\mathcal{G}$, is it possible to define a local cone approximation K such that a function f is G-differentiable if and only if f is K-differentiable, with $p(\cdot,\cdot)$ as the K-derivative of f, i.e. $f^K = p$?

The local cone approximation K we must find to answer question (*b*), will not be defined on $2^{\mathbb{R}^n} \times \mathbb{R}^n$, as it makes no sense to consider a derivative at a point not belonging to the graph of f. This cone approximation will be therefore defined on $2^Y \times Y'$, with

$$Y \quad = \{\text{epi}f : f \text{ is } G\text{-differentiable}\}$$
$$Y' \quad = \{x \in \mathbb{R}^{n+1} : x \in \text{graph}f,\ f \text{ is } G\text{-differentiable}\}.$$

Moreover, since we are interested only the approximations of epigraphs of functions, it is clear that the property (*e*) of Definition 1 must be replaced by the following axiom

(*e bis*) $\quad K(\varphi_\Psi(\text{epi}f), \varphi_\Psi(x, f(x)) = \varphi_\Psi\left[K(\text{epi}f, (x, f(x))\right],$

for every linear $\Psi : \mathbb{R} \to \mathbb{R}$ and $\varphi_\Psi(x, f(x)) = (x, \Psi(f(x)),$ as we consider only linear homeomorphisms which transform epigraphs in to epigraphs.

Pappalardo (1992), by means of two examples proves that, in general, the two questions (*a*) and (*b*) have a negative answer. However, under suitable, rather mild, conditions the two approaches are equivalent.

Theorem 17 (Pappalardo (1992))

(17.1) Consider a local cone approximation K such that for every function f which is K-differentiable at a point $\overline{x} \in \mathbb{R}^n$, we have

$$\lim_{y \to 0} \frac{f(x^0 + y) - f(x^0) - f^K(x^0, y)}{\|y\|} = 0.$$

Then there exists a subset G of $\mathcal{G}$ such that, for every function which is K-differentiable, f^K is the G-derivative of f.

(17.2) Consider a set $G \subset \mathcal{G}$ with the following properties

$$p \in G \to \alpha p \in G, \forall \alpha \in \mathbb{R}.$$

Then there exists a local cone approximation K such that for every function f which is G-differentiable, $p(\cdot,\cdot)$ is the K-derivative of f.

Pappalardo (1992) studies also the relations between K-derivatives and G-semiderivatives.

Another approach to generalized differentiability, quite similar to the approach of Elster and Thierfelder, is due to D. Ward (1988), by means of the notion of "quantificational tangent cone" ("q-cone"). This definition contains, as particular cases, all the axiom of Elster and Thierfelder. Ward gives a definition of generalized directional derivative which is the same of the one given by Elster and Thierfelder.

For a discussion of constraint qualifications in the context of the q-cones, see Merkovsky and Ward (1990).

Another general approach to differentiability, suitable for applications to optimization theory, has been elaborated by S. Komlosi (1993) and subsequently generalized by Komlosi and Pappalardo (1994).

Definition 11

The bifunction $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a *first order approximation* of the function $f : \mathbb{R}^n \to \mathbb{R}$ at the point x, if:

(i) $h(x, y)$ is linearly homogeneous in y;

(ii) $D_f^h(x) \subset D_f(x)$, where

$$D_f^h(x) = \{y \in \mathbb{R}^n : h(x, y) < 0\},$$
$$D_f(x) = \{y \in \mathbb{R}^n : \exists T > 0, \forall t \in (0, T), f(x + ty) - f(x) < 0\}.$$

The sets $D_f^h(x)$ and $D_f(x)$ are cones; $D_f(x)$ is the *cone of descent directions* of f at x.

It is easy to see that the usual directional derivative $f'(x, y)$, the Clarke generalized directional derivative $f^o(x, y)$ and the Dini upper directional derivative $f_+^D(x, y)$ are all examples of first order approximations of f at x.

Also Ioffe (1979, 1984) has introduced a definition of first order approximation, which is, however, more restrictive than the one of Definition 11, since this author requires, as in Definition 11, the positive homogeneity of the approximation h, but instead of condition (ii), he requires the condition

$(ii)'$ $h(x, y) \geq f_+^D(x, y), \forall y \in \mathbb{R}^n$.

See also Jeyakumar (1987).

By means of Definition 11, Komlosi obtains necessary optimality conditions for $(\mathcal{P}_1)$ and for $(\mathcal{P})$; for this last case Komlosi introduces a regularity condition, which is in fact a constraint qualification.

The concept of first order approximation of a function has been subsequently generalized by Komlosi and Pappalardo (1994), in view of obtaining general optimality conditions for $(\mathcal{P}_1)$ and $(\mathcal{P})$.

Let us consider $(\mathcal{P}_1)$ and consider the strict lower level set:

$$L_f(x^0) = \left\{x \in \mathbb{R}^n : f(x) < f(x^0)\right\}, \ x^0 \in C.$$

It is evident that x^0 is a solution of $(\mathcal{P}_1)$ if and only if

$$L_f(x^0) \cap C = \emptyset,$$

and that x^0 is a local solution of $(\mathcal{P}_1)$ when

$$L_f(x^0) \cap C \cap N(x^0) = \emptyset,$$

where $N(x^0)$ is an appropriate neighborhood of x^0.

The verification of the previous conditions is, however, in general rather complicate. In order to get more tractable optimality conditions, it is better to approximate the sets $L_f(x^0)$ and C by other sets having a simpler structure, e.g., the local cone approximations of Elster and Thierfelder .

Definition 12

An ordered pair (K_1, K_2) of local cone approximations is *admissible* for $(\mathcal{P}_1)$, when the condition

$$K_1\left(L_f(x^0), x^0\right) \cap K_2(C, x^0) = \emptyset$$

is a necessary optimality condition for $x^0 \in C$ to be a local optimal solution of $(\mathcal{P}_1)$.

The following result shows how to build pairs of local cone approximations admissible for $(\mathcal{P}_1)$. We recall that a local cone approximation K is *isotone*, when

$$S \subset T, x^0 \in \mathrm{cl}(S) \cap \mathrm{cl}(T) \Rightarrow K(S, x^0) \subset K(T, x^0).$$

Theorem 18 (Komlosi and Pappalardo (1994))

Let $K_1(M, x^0)$ be an isotone local cone approximation and set

$$K_2(M, x^0) = \mathbb{R}^n \backslash K_1(\mathbb{R}^n \backslash M, x^0).$$

Then the pair (K_1, K_2) is admissible for $(\mathcal{P}_1)$.

Definition 13

A bifunction $h(x^0, y)$, linearly homogeneous in y, is a generalized first order approximation of $f(x)$ at x^0, when there exists an admissible pair (K_1, K_2) of local cone approximations for $(\mathcal{P}_1)$ such that

$$D_h(x^0) \subset K_1(L_f(x^0), x^0),$$

where $D_h(x^0) = \{y \in \mathbb{R}^n : h(x^0, y) < 0\}$.

It is easily seen that this concept of first order approximation is more general then the one of Definition 11. It is sufficient to choose $K_1(S, x^0) = Z(S, x^0)$ and $K_2(S, x^0) = F(S, x^0)$ to see that the properties requested by Definition 11 are satisfied.

By means of this new definition, Komlosi and Pappalardo (1994) obtain necessary optimality conditions for $(\mathcal{P}_1)$. Rocca (1995) extends this approach to problem $(\mathcal{P})$. It is also possible to obtain relationships between K-derivatives, G-semiderivatives and first order approximations. This has been (partially) done by Komlosi and Pappalardo (1994). By completing their analysis, it is seen that indeed the concept of generalized first order approximation is one of the most general "containers" of the various concepts of generalized directional derivatives considered in the literature.

REFERENCES

K. J. ARROW, L. HURWICZ, H. UZAWA (1961), Constraint qualifications in maximization problems, Naval Research Logistics Quarterly, 8, 175-191.

J. ABADIE (1967), On the Kuhn-Tucker theorem; in J. Abadie (Ed.), Nonlinear Programming, North Holland, Amsterdam, 19-36.

F. H. CLARKE (1983), Optimization and Nonsmooth Analysis, J. Wiley & Sons, New York.

K. H. ELSTER, J. THIERFELDER (1985), The general concept of cone approximations in nondifferentiable optimization; in V.F. Demyanov, D. Pallaschke (Eds.), Nondifferentiable Optimization. Motivations and Applications, Springer Verlag, Berlin, 170-189.

K. H. ELSTER, J. THIERFELDER (1988a), Abstract cone approximations and generalized differentiability in nonsmooth optimization, Optimization, 19, 315-341.

K. H. ELSTER, J. THIERFELDER (1988b), On cone approximations and generalized directional derivatives; in F.H. Clarke, V. F. Demyanov, F. Giannessi (Eds.), Nonsmooth Optimization and Related Topics, Plenum Press, New York, 133-154.

F. GIANNESSI (1989), Semidifferentiable functions and necessary optimality conditions, J. Optim. Theory Appl., 60, 191-241.

F. GIANNESSI (2005), Constrained Optimization and Image Space Analysis. Volume I: Separation of Sets and Optimality Conditions, Springer, New York.

G. GIORGI, A. GUERRAGGIO (1992), On a characterization of Clarke's tangent cone, J. Optim. Theory Appl., 74, 369-372.

G. GIORGI, A. GUERRAGGIO, J. THIERFELDER (2004), Mathematics of Optimization. Smooth and Nonsmooth Case, Elsevier, Amsterdam.

F. J. GOULD, J. W. TOLLE (1971), A necessary and sufficient qualification for constrained optimization, SIAM J. Applied Math. 20, 164-172.

M. GUIGNARD (1969), Generalized Kuhn–Tucker conditions for mathematical programming problems in a Banach space, SIAM J. Control, 7, 232-241.

M. R. HESTENES (1966), Calculus of Variations and Optimal Control Theory, J. Wiley, New York.

M. R. HESTENES (1975), Optimization Theory: The Finite Dimensional Case, J. Wiley, New York.

A.D. IOFFE (1979), Necessary and sufficient conditions for a local minimum: I, SIAM Journal on Control and Optimization, 17, 245-250.

A.D. IOFFE (1984), Calculus of Dini subdifferentials of functions and contingent derivatives of set-valued maps, Nonlinear Analysis 8, 517-539.

V. JEYAKUMAR (1987), On optimality conditions in nonsmooth inequality constrained minimization, Numer. Funct. Anal. Optim., 9, 535-546.

S. KOMLOSI (1993), Quasiconvex first-order approximations and Kuhn-Tucker type optimality conditions, European J. Op. Res. 65, 327-335.

S. KOMLOSI, M. PAPPALARDO (1994), A general scheme for first order approximations in optimization, Optimization Methods and Software, 3, 143-152.

H. W. KUHN, A. W. TUCKER (1951), Nonlinear programming; in J. Neyman (Ed.), Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, 481-492.

P. MICHEL, J.-P. PENOT (1984), Calcul sous-différentiel pour des fonctions Lipschitziennes et non-Lipschitziennes, Comptes Rendus des Séances de l'Académie des Sciences Paris, Séries I, Mathématique, 298, 269-272.

R. R. MERKOVSKY, D. E. WARD (1990), General constraint qualifications in nondifferentiable programming, Math. Programming 47, 389-405.

M. PAPPALARDO (1992), On Semidifferentiable Functions, Optimization, 24, 207-217.

M. PAPPALARDO (1993a), Semidifferentiability and extremum problems; in K. H. Elster, C. Sutti (Eds.), Mathematical Optimization Theory, Methods and Applications, Libreria Universitaria Editrice, Verona, 77-84.

M. PAPPALARDO (1993b), Some calculus rules for semidifferentiable functions and related topics; in F. Giannessi (Ed.), Nonsmooth Optimization Methods and Applications, Gordon and Breach Science Publishers, New York, 281-294.

M. PAPPALARDO, A. UDERZO (1999), G-semidifferentiability in euclidean spaces, J. Optim. Theory Appl., 101, 221-229.

S. M. ROBINSON (1987), Local structure of feasible sets in nonlinear programming, Part 3: stability and sensitivity, Mathematical Programming Study 30, 45-66.

M. ROCCA (1994), Funzioni G-semiderivabili: condizioni di ottimalità e di qualificazione dei vincoli, Atti del Diciottesimo Convegno A.M.A.S.E.S., Modena, 5-7 settembre 1994, 509-525.

M. ROCCA (1995), Approssimazioni generalizzate del primo ordine, Atti XIX Convegno A.M.A.S.E.S., Cacucci Editore, Bari, 561-575.

R. T. ROCKAFELLAR (1970), Convex Analysis, Princeton University Press, Princeton, New York.

R. T. ROCKAFELLAR (1980), Generalized directional derivatives and subgradients of nonconvex functions, Canad. J. Math., 32, 257-280.

R. T. ROCKAFELLAR (1981), The Theory of Subgradients and Its Applications to Problems of Optimization: Convex and Nonconvex Functions, Heldermann Verlag, Berlin.

C. URSESCU (1982), Tangent set's calculus and necessary conditions for extremality, SIAM J. Control Optim., 20, 563-574.

N. D. YEN (1995), On G-semidifferentiable functions in euclidean spaces, J. Optim. Theory Appl., 85, 377-392.

D. WARD (1988), The quantificational tangent cones, Can. J. Math., 40, 666-694.