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Proofs of the Hawkins-Simon Conditions and Other Related Topics

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Abstract. In the first part of the paper we provide a survey (not exhaustive) of various "chains of proofs" of the Hawkins-Simon conditions for the solvability of the classical "open" Leontief input-output model. In the second part of the paper we take into consideration some variants and generalizations of the Hawkins-Simon conditions. In particular, we discuss possible generalizations of the said conditions to linear joint productions models.

Key words. Hawkins-Simon conditions, M-matrices, linear joint production models.

1. Introduction

The celebrated (at least among economists) *Hawkins-Simon conditions* refer to a result, usually attributed to D. Hawkins and H. A. Simon (1949), that gives necessary and sufficient conditions for the existence of a nonnegative output vector which solves the equilibrium relations in an "open" Leontief input-ouput model. More precisely, the said result states necessary and sufficient conditions such that a system of the type

$$\sum_{j=1}^{n} a_{ij}x_j + c_i = x_i, \quad i = 1, ..., n,$$

where $a_{ij} \geqq 0$ is the amount of the i-th good used to produce one unit of the j-th good, $x_j \geqq 0$ is the amount of the j-th good produced by the economic system and $c_i \geqq 0$ is the amount of final demand for good i, has a nonnegative solution x_i, $i = 1, ..., n$. If we define the square matrix B, of order n, given by $B = I - A$, where I is the identity matrix and $A = [a_{ij}]$, $i, j = 1, ..., n$, then one of the most quoted versions of the *Hawkins-Simon Theorem* states that the following two statements are equivalent:

(i) There exists a nonnegative vector $x \in \mathbb{R}^n$ such that $Bx > [0]$. See further for the notations on the comparison of vectors of $\mathbb{R}^n$.

(ii) All the successive "leading principal minors" of B are positive, that is,

$$b_{11} > 0, \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} > 0, ..., |B| > 0.$$

Note that $b_{ij} \leqq 0$, $\forall i \neq j$. The square matrices with this property are usually called *Z-matrices* or *matrices of class Z*.

From (ii) it appears that the *k-th order leading principal minor* or *k-th order North-West principal minor* of a square matrix A, of order n, denoted also by Δ_k or $\Delta_k(A)$, is the determinant of the square submatrix of A, of order k, consisting of the *first* k rows and *first* k columns of A. Usually, the above conditions (ii) are known as *Hawkins-Simon conditions,* even if this is not completely correct (see further).

A *k-th order principal minor* of the square matrix A, of order n, denoted also by $\tilde{\Delta}_k$ or $\tilde{\Delta}_k(A)$, is a determinant obtained from A by considering its k rows and the *corresponding* k columns. In other words, if we consider a *permutation matrix* P, a k-th order principal minor of A, is the k-th order leading principal minor of $PAP^\top$, where P is some permutation matrix. It can be proved that there are

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$$

possible k-th order principal minors of A, square of order n.

In the present paper we denote by $[0]$ the zero vector of $\mathbb{R}^n$ and also the *zero matrix* of order (m, n). if $x \in \mathbb{R}^n$, then we say that

- x is *nonnegative* if $x_i \geqq 0$, $i = 1, ..., n$; we write $x \geqq [0]$.
- x is *semipositive* if $x \geqq [0]$, but $x \neq [0]$; we write $x \geq [0]$.
- x is *positive* if $x_i > 0$, $i = 1, ..., n$; we write $x > [0]$.

The same conventions and notations are used to compare a matrix A of order (m, n) with the zero matrix $[0]$, of the same order:

- A is *nonnegative* if $a_{ij} \geqq 0$, $i = 1, ..., m$; $j = 1, ..., n$. We write $A \geqq [0]$.
- A is *semipositive* if $A \geqq [0]$, but $A \neq [0]$. We write $A \geq [0]$.
- A is *positive* if $a_{ij} > 0$, $i = 1, ..., m$; $j = 1, ..., n$. We write $A > [0]$.

We recall that a square matrix A, of order n, is said to be *decomposable* or *reducible,* if there exists a permutation matrix P such that

$$PAP^\top = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with A_{11} square submatrix of A (and hence also A_{22} square) and at least one of the submatrices A_{12}, A_{21} a zero matrix. If A is not decomposable, it is said to be *indecomposable* or *irreducible.*

We recall that, given $A \geqq [0]$, square of order n, its *Frobenius eigenvalue* or *dominant root,* denoted by $\lambda^*(A)$, is that nonnegative (real) root of the characteristic equation of A, such that

$$\lambda^*(A) \geqq |\lambda|,$$

being λ any other eigenvalue of A.

The Hawkins-Simon conditions are a part of a more general theory, usually referred as the theory of (nonsingular) M-*matrices*, called also by some authors, K-*matrices.* Given the (square) Z-matrix B, i. e. $B = [b_{ij}]$, $b_{ij} \leqq 0$, $\forall i \neq j$, B is an M-matrix if B satisfies the Hawkins-Simon conditions (ii) or any one of a long list of equivalent conditions. See. e. g., Bapat and Raghavan (1997), Berman and Plemmons (1994), Fiedler anf Ptàk (1962), Giorgi (2022), Magnani and Meriggi (1981), Murota and Sugihara (2022), Plemmons (1977), Poole and Boullion (1974). Given the Z-matrix B, there are more than 60 conditions equivalent to the Hawkins-Simon conditions! However, the "chains of proofs" of these conditions are often not focused on possible economic interpretations or economic applications of the same, hence the Hawkins-Simon conditions are only one ring of the said "chains". For example, Fiedler and Ptàk (1962) give a complete proof of 13 equivalent conditions, performing an elegant and stringent chain of proofs, where, however, the Hawkins-Simon conditions are put into relation with other mathematical conditions, with doubtful economic meaning. The aim of the present paper is to take into consideration various "chains of proofs" of the Hawkins-Simon conditions, which are related to those mathematical properties having some utility or meaning in the analysis of linear multi-sectoral economic models.

The paper is organized as follows. Section 2 contains a survey (not exhaustive) of various "chains of proofs" of the Hawkins-Simon conditions and of some related equivalent conditions. The final Section 3 contains some remarks on possible variants and/or generalizations of the Hawkins-Simon conditions. In particular, we discuss possible generalizations of the said conditions to linear joint production models.

2. Proofs of the Hawkins-Simon conditions

A) **The chain of proofs of Nikaido (1968, 1970) and some extensions.**

See also Kemp and Kimura (1978), Murata (1977), Takayama (1985) and Woods (1978). Let B be a square matrix of order n, such that $b_{ij} \leqq 0$, $\forall i \neq j$, that is B is a Z-*matrix* or a *matrix of class* Z. Nikaido (1968, 1970) takes into consideration the following equivalent conditions.

(I) For some $c > [0]$ the system $Bx = c$ has a solution $x \geqq [0]$. Equivalently: there exists an $x \geqq [0]$ such that $Bx > [0]$.

(II) For any $c \geqq [0]$, there exists an $x \geqq [0]$ such that $Bx = c$.

(III) The leading principal minors of B are all positive:

$$\Delta_1 = b_{11} > 0;\ \Delta_2 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0;\ \Delta_3 = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} > 0; ...; \Delta_n = |B| > 0.$$

In addition to the above conditions of Nikaido, we consider also the following condition:

(IV) All principal minors of B are positive:

$$\tilde{\Delta}_1(B) > 0,\ \tilde{\Delta}_2(B) > 0, ..., \tilde{\Delta}_n(B) = |B| > 0.$$

Nikaido calls (I) the *weak solvability condition* and (II) the *strong solvability condition.* Hence, we say that the system $Bx = c$ is *weakly solvable* if (I) holds and *strongly solvable* if (II) holds.

Condition (III) is usually attributed to Hawkins and Simon (1949), even if it was considered by Georgescu-Roegen (1951, 1966). More correctly, the conditions considered by Hawkins and Simon are conditions (IV), previously discovered by Ostrowski (1937) and studied also by Kotelianski (1952). Indeed, Gantmacher (1959) calls (IV) "conditions of Kotelianski". Furthermore, it seems that these conditions, as well as several results related to the Perron-Frobenius Theorem, were discovered at the beginning of the last century by the French mathematician M. Potron. See Bidard (2007).

A square matrix (not necessarily a Z-matrix) of order n, which satisfies condition (IV) is said to be a *P-matrix*. See Fiedler and Ptàk (1962), Nikaido (1968).

Theorem 1. Given the Z-matrix B, of order n, the conditions $(I), (II), (III)$, and (IV) are equivalent.

Proof.

The implications $(II) \implies (I)$ and $(IV) \implies (III)$ are trivial. We prove the implication

$$(I) \implies (III).$$

We perform the proof by induction on n, the order of B. Obviously the implication is true for $n = 1$, so it must be $b_{11} > 0$. Now we suppose that the implication is true for $(k-1)$ and we prove that then it holds true for k. Suppose the system $Bx = c$, with k equations, to be weakly solvable. The first equation of the system can be written in the form

$$b_{11}x_1 + b_{12}x_2 + ... + b_{1k}x_k = c_1,$$

from which

$$b_{11}x_1 = c_1 - \sum_{j=2}^{k} b_{ij}x_j > 0,$$

as $c > [0]$ and hence $c_1 > 0$, $b_{ij} \leqq 0$, $\forall i \neq j$, and $x_j \geqq 0$, by the assumption of weak solvability. From these conditions it follows $b_{11} > 0$. This allows to divide by b_{11} and to transform the system into an equivalent system of the following type

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ 0 & \tilde{b}_{22} & \cdots & \tilde{b}_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \tilde{b}_{k2} & \cdots & \tilde{b}_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} c_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_k \end{bmatrix},$$

where

$$\tilde{b}_{ij} = b_{ij} - \frac{b_{i1}}{b_{11}} b_{1j} \leqq 0, \ i \neq j,$$

and

$$\tilde{c}_i = c_i - \frac{b_{i1}}{b_{11}} c_1 > 0, \ i = 1, ..., k.$$

The following system of $(k-1)$ equations

$$\begin{bmatrix} \tilde{b}_{22} & \cdots & \tilde{b}_{2k} \\ \vdots & \cdots & \vdots \\ \tilde{b}_{k2} & \cdots & \tilde{b}_{kk} \end{bmatrix} \begin{bmatrix} x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} \tilde{c}_2 \\ \vdots \\ \tilde{c}_k \end{bmatrix}$$

is weakly solvable and we can apply the induction hypothesis: the matrix $\tilde{B} = \left[\tilde{b}_{ij}\right]$, $i, j = 2, ..., k$, verifies (III). But the leading principal minors of B are given, on the grounds of a well known property of determinants, by the correspondent leading principal minors of $\tilde{B}$, multiplied by $b_{11} > 0$. So, we have $\Delta_k(B) = b_{11}\Delta_{k-1}(\tilde{B}) > 0$ and the implication is proved.

Now we prove the implication

$$(III) \implies (II).$$

Also this implication is proved by induction on the number of equations of the system. The implication is true for $n = 1$. Indeed, if $b_{11}x_1 = c_1$, with $b_{11} > 0$ and $c_1 \geqq 0$, obviously there exists a solution $x_1 = \frac{c_1}{b_{11}} \geqq 0$. Let us suppose that the implication holds for a system with $(k-1)$ equations and let us prove that then it holds also for a system with k equations. As performed in the previous proof, we consider the following system of $(k-1)$ equations

$$\tilde{B} \begin{bmatrix} x_2 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} \tilde{c}_2 \\ \vdots \\ \tilde{c}_k \end{bmatrix}.$$

Owing to the relations existing between the minors of B and of $\tilde{B}$, and in particular from the fact that

$$\det \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \cdots & \vdots \\ b_{r1} & \cdots & b_{rr} \end{pmatrix} = b_{11} \det \begin{pmatrix} \tilde{b}_{22} & \cdots & \tilde{b}_{2r} \\ \vdots & \cdots & \vdots \\ \tilde{b}_{r2} & \cdots & \tilde{b}_{rr} \end{pmatrix}$$

and that $b_{11} > 0$ and

$$\det \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ldots & \vdots \\ b_{r1} & \cdots & b_{rr} \end{pmatrix} > 0,$$

we deduce that

$$\det \begin{pmatrix} \tilde{b}_{22} & \cdots & \tilde{b}_{2r} \\ \vdots & \ldots & \vdots \\ \tilde{b}_{r2} & \cdots & \tilde{b}_{rr} \end{pmatrix} > 0,$$

since $\tilde{B}$ verifies (III). We apply the induction hypothesis: there exists

$$\tilde{x} = \begin{pmatrix} x_2 \\ \vdots \\ x_k \end{pmatrix} \geqq [0]$$

such that for any $\tilde{c} \geqq [0]$ the system $\tilde{B}\tilde{x} = \tilde{c}$ has a solution. Moreover, we have $x_1 \geqq 0$, being

$$x_1 = \frac{1}{b_{11}} \left[c_1 - \sum_{j=2}^{k} b_{1j} x_j \right] \geqq 0.$$

Also this implication is proved.

Finally, we prove the implication $(II) \implies (IV)$, i. e. (II) implies that B is a P-matrix. Since the implication $(IV) \implies (III)$ is trivial, we have shown the equivalence $(III) \iff (IV)$. We follow Woods (1978). Assume that (II) is satisfied. Let P be a *permutation matrix.* From

$$Bx = c, \; x \geqq [0] \, , \; c \geqq [0] \, ,$$

we have

$$PBx = PBP^{-1}Px = Pc$$

or

$$PBP^{-1}y = d,$$

with $y = Px$, $d = Pc$. By this device, any principal submatrix of B can be transformed into a leading principal submatrix of PBP^{-1}. Note that if B satisfies the sign conditions, i. e. $B \in Z$, then so does PBP^{-1}. Also, if $c \geqq [0]$, then $d \geqq [0]$. Condiitons (II) and (III) are equivalent. So, assuming (II) is satisfied, for $Bx = c$, it will be satisfied for $PBP^{-1}y = d$, where P is an arbitrary permutation matrix. Then PBP^{-1} satisfies condition (III). As this is true for an arbitrary permutation matrix, ir follows that all principal minors of B are positive, i. e. B is a P-matrix. □

Remark 1. On the grounds of the proof of the last part of Theorem 1, we can remark (see Giorgi (2022), Magnani and Meriggi (1981)) that the class of (nonsingular) M-matrices (or K-matrices) verifies the following properties:

1) The Z-matrix B is an M-matrix if and only if $B^\top$ is an M-matrix.

2) The Z-matrix B is an M-matrix if and only if PBP^{-1} is an M-matrix, being P any permutation matrix.

3) The Z-matrix B is an M-matrix if and only if DBE is an M-matrix, with D and E diagonal matrices, with all positive diagonal elements.

The above transformations may be useful for further developments: for instance, transformation 1) allows to rewrite some propositions concerning the solvability of a linear economic model, in their dual form, i. e. referred to $B^\top$, which can be useful, for instance, to establish the equivalence between *productivity* and *profitability* of linear economic models, with no joint productions. Transformation 2), besides its role on proving the equivalence between conditions (III) and (IV), allows to refer some properties of a decomposable Z-matrix to the main blocks of its "Gantmacher normal form". See Gantmacher (1959), Giorgi and Magnani (1978). Transformation 3) is useful, for instance, in proving that a Metzler matrix A (i. e. a square matrix A such that $-A \in Z$, that is $a_{ij} \geqq 0$, $\forall i \neq j$) is *stable,* regardless of the choice of the positive speeds of adjustment in a competitive Walrasian economic model. See, e. g., Woods (1978).

One of the most useful (in economic applications) characterizations of the class of (nonsingular) M-matrices is :"the inverse of the Z-matrix B esists and is semipositive, i. e. $B^{-1} \geq [0]$". Note that B^{-1} has all semipositive lines. We now prove the following result.

Theorem 2. Given the Z-matrix B, of order n, then the equivalent conditions $(I), (II), (III)$ and (IV), are in turn equivalent to:

(V) B^{-1} exists and it holds $B^{-1} \geq [0]$.

Proof. First we prove the implication $(II) \implies (V)$. Assume (II), that is the system $Bx = c$ has a nonnegative solution for any nonnegative vector c, hence, in particular, for the i-th unit vector e^i, $i = 1, ..., n$. Let us denote by x^i the nonnegative solution of $Bx = e^i$. Then every x^i must be semipositive. Hence $B^{-1} = \left[x^1, ..., x^n\right] \geq [0]$. We finish the proof by observing that the reverse implication is obvious. □

Remark 2. If B is *indecomposable,* statements (II) and (V) can be given in a modified form, respectively:

$(II)'$: For any $c \geqq [0]$ there exists an $x > [0]$ such that $Bx = c$.

$(V)'$: B^{-1} exists and it holds $B^{-1} > [0]$.

See, e. g., Kemp and Kimura (1978).

In order to introduce another condition, equivalent to the previous ones $(I), (II), (III), (IV)$ and (V), we need the following notion.

Definition 1. A square (real) matrix A, of order n, is said to have a *quasi-dominant diagonal* (in the sense of McKenzie (1960)), if there exist scalars $d_i > 0$, $i = 1, ..., n$, such that

$$d_i |a_{ii}| > \sum_{j \neq i} d_j |a_{ij}|, \quad i = 1, ..., n.$$

If, in addition, $a_{ii} > 0$, $i = 1, ..., n$, then A is said to have a *positive* quasi-dominant diagonal. Similarly, A has a *negative* quasi-dominant diagonal if $a_{ii} < 0$, $i = 1, ..., n$, and the above inequalities hold.

Remark 3. The above definition takes into consideration the *rows* of matrix A. It is also possible to take into consideration the *columns* of A, by saying that A has a *column quasi-dominant diagonal* if there exist $d_j > 0$, $j = 1, ..., n$, such that

$$d_j |a_{jj}| > \sum_{i \neq j} d_i |a_{ij}|, \quad j = 1, ..., n.$$

However, the two definitions are *equivalent* and hence the distinction between row quasi-dominant diagonal and column quasi-dominant diagonal is superfluous. This is true for the above definition, due to McKenzie (1960), but not for other definitions. See Kemp and Kimura (1978), Takayama (1986). The notion of matrices with a quasi-dominant diagonal is very useful in several economic applications, as shown by McKenzie (1960) himself. The following two results are due to this author.

Theorem 3. Let A be a square matrix of order n.

(i) If A has a quasi-dominant diagonal, then it is nonsingular.

(ii) If A has a positive quasi-dominant diagonal, then all its principal minor are positive, i. e. A is a P-matrix.

Remark 4. Proposition (i) of Theorem 3 is a generalization of a classical result of the French mathematician J. Hadamard, who established in 1903 this result for matrices having a more restrictive diagonal dominance property than the one introduced by McKenzie. Indeed, the usual definition in the literature, due to Hadamard, is that A has a dominant diagonal if

$$|a_{jj}| > \sum_{i \neq j} |a_{ij}| \text{ for all } j,$$

or if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \text{ for all } i.$$

The two above definitions are *not* equivalent. If D is a diagonal matrix, with positive diagonal elements, then the definition of McKenzie says that DA (or equivalently AD) has a dominant diagonal in the Hadamard sense.

Proposition (ii) may be used (see, e. g., Kemp and Kimura (1978)) to give an alternative proof of the implication $(I) \implies (III)$. Another result of McKenzie we take into consideration, characterizes in fact condition (II) in terms of matrices with a quasi-dominant diagonal.

Theorem 4. Let $B \in Z$, B of order n. Then, the equivalent conditions $(I), (II), (III), (IV)$ and (V) are in turn equivalent to:
(VI) B has a positive quasi-dominant diagonal.

Proof (McKenzie). Following McKenzie (1960), we prove the equivalence

$$(II) \iff (VI).$$

(a) Suppose that there exist $d_i > 0$, $i = 1, ..., n$, such that B has a positive quasi-dominant diagonal. Then B is nonsingular, by Theorem 3, hence a (unique) solution of (II) exists. To show that $x \geqq [0]$, suppose that $x_j < 0$ for $j \in J \neq \varnothing$ and $x_j \geqq 0$ for $j \notin J$, where J is a set of indices. Consider

$$\sum_{j \notin J} b_{ij}x_j + \sum_{j \in J} b_{ij}x_j = c_i \geqq 0, \text{ for } i \in J.$$

Multiplying by d_i and summing, we obtain

$$\sum_{i \in J}\sum_{j \notin J} d_i b_{ij} x_j + \sum_{i \in J}\sum_{j \in J} d_i b_{ij} x_j = \sum_{i \in J} d_i c_i \geqq 0. \tag{1}$$

Clearly, the first term on the left is nonpositive, since $x_j \geqq 0$ for $j \notin J$ and $b_{ij} \leqq 0$, for $i \neq j$. By assumption of positive quasi-dominant diagonal, we have

$$\sum_{i \in J,\ i \neq j} d_i \left| b_{ij} \right| < d_j \left| b_{ij} \right|$$

for all j, hence for $j \in J$. Since $b_{jj} > 0$, this implies

$$\sum_{i \in J,\ i \neq j} d_i b_{ij} + d_j b_{jj} = \sum_{i \in J} d_i b_{ij} > 0$$

for $j \in J$. Hence $\sum_{i \in J}\sum_{j \in J} d_i b_{ij} x_j < 0$, that is the second term in the left-hand side of (1) is negative. Thus the left-hand side of (1) is negative, which is a contradiction.

(b) Consider $Bx = c$. By assumption, for any $c \geqq [0]$ there exists a solution $x \geqq [0]$. In particular, let $c > [0]$. Then $x > [0]$ if $b_{ii} > 0$ for all i, being $b_{ij} \leqq 0$, $\forall i \neq j$. Hence $B^\top$ has a positive quasi-dominant diagonal realized by this vector x. Then by the above, $B^\top p = \pi$ has a unique solution $p \geqq [0]$. In particular, let $\pi > [0]$, then $p > [0]$, since $b_{ii} > 0$ for all i and $b_{ij} \leqq 0$, $i \neq j$. But this means that also B has a positive quasi-dominant diagonal with respect to this vector p. $\square$

In order to introduce the next condition of the present "chain of proofs", we first observe that, given a square matrix $B \in Z$, of order n, it is always possible to write the same in the form

$$B = \rho I - A,$$

with $A \geqq [0]$, A square of order n, and with $\rho \in \mathbb{R}$. Indeed, it is sufficient to choose ρ such that

$$\rho \geqq \max_i \{b_{ii}\},$$

and then choose the matrix

$$A = \rho I - B$$

which is nonnegative by construction. The expression $B = \rho I - A$ is called a "split" or "splitting", and this form often appears in the analysis of several economic models (but not only in these applications!). One of the many characterizations of (nonsingular) M-matrices is:

• $B \in Z$, square of order n, is an M-matrix if and only if, when written in the form $B = \rho I - A$, with $A \geqq [0]$, A square of order n, it results $\rho > \lambda^*(A)$.

On the grounds of the previous result, therefore we can state the following theorem.

Theorem 5. Let be given the Z-matrix $B = \rho I - A$, with $A \geqq [0]$, square of order n. Then, the equivalent conditions $(I), (II), (III), (IV), (V)$ and (VI) are in turn equivalent to:

(VII) It holds $\rho > \lambda^*(A)$.

Usually Theorem 5 is given as a corollary of the celebrated Theorem of Perron-Frovebius, by showing that (VII) is equivalent to $(\rho I - A)^{-1} \geq [0]$. See, e. g., Morishima (1964), Nikaido (1968, 1970), Pasinetti (1977), Schwartz (1961), Takayama (1985). This result is often proved by means of a property of convergence of a suitable series of matrices ("C. Neumann series"). A direct and elegant proof which avoids the said series is given by Debreu and Herstein (1953). We follow this proof. First we need a simple preliminary result.

Lemma 1. Let $A \geqq [0]$ be of order n, with $\lambda^*(A)$ its Frobenius root. If for an $x > [0]$, $Ax \leqq \rho x$ (resp. $\geqq \rho x$), then $\lambda^*(A) \leqq \rho$ (resp. $\lambda^*(A) \geqq \rho$). If for an $x \geqq [0]$, $Ax < \rho x$ (resp. $Ax > \rho x$), then $\lambda^*(A) < \rho$ (resp. $\lambda^*(A) > \rho$).

Proof. As the proofs of the above statements are practically identical, we present only the proof of the first one. Let $x^0 \geq [0]$ be a characteristic vector of $A^\top$, associated with $\lambda^*(A) : A^\top x^0 = \lambda^*(A) x^0$. $Ax \leqq \rho x$ with $x > [0]$, therefore $(x^0)^\top Ax \leqq \rho (x^0)^\top x$, i. e. $\lambda^*(A)(x^0)^\top x \leqq \rho (x^0)^\top x$, and, since $(x^0)^\top x > 0$, $\lambda^*(A) \leqq \rho$. □

Proof of Theorem 5. We prove that $(\rho I - A)^{-1} \geq [0]$ if and only if $\rho > \lambda^*(A)$.

Sufficiency. Since $\rho > \lambda^*(A)$, the system

$$(\rho I - A)x = y \tag{2}$$

has a unique solution $x = (\rho I - A)^{-1}y$ for every y; we show that $y \geqq [0]$ implies $x \geqq [0]$.

If x had negative components (2) could be given in the form (by permutations of the rows and of the corresponding columns)

$$\begin{bmatrix} \rho I - A_{11} & -A_{12} \\ -A_{21} & \rho I - A_{22} \end{bmatrix} \begin{bmatrix} -x^1 \\ x^2 \end{bmatrix} = y,$$

where $x^1 > [0]$, $x^2 \geqq [0]$, $y \geqq [0]$. Therefore $-(\rho I - A_{11})x^1 - A_{12}x^2 \geqq [0]$, i. e. $-(\rho I - A_{11})x^1 \geqq [0]$, i. e. $A_{11}x^1 \geqq \rho x^1$. From Lemma 1 the Frobenius root of A_{11}, $\lambda^*(A_{11})$, is such that $\lambda^*(A_{11}) \geqq \rho$, a contradiction to the fact that $\lambda^*(A) \geqq \lambda^*(A_{11})$ and $\rho > \lambda^*(A)$.

Necessity. Since $(\rho I - A)^{-1} \geq [0]$, to a $y > [0]$ corresponds an $x \geqq [0]$. Therefore from $\rho x - Ax = y$ follows $Ax < \rho x$ and, by Lemma 1, $\lambda^*(A) < \rho$. □

Remark 5. It must be remarked that if A is *indecomposable,* then $(\rho I - A)^{-1} > [0]$ if and only if $\rho > \lambda^*(A)$.

B) **A variant of the previous chain of proofs.**

By exploiting the theory of M-matrices, it is possible to consider other variants of the previous chain of proofs. For example, we can state the following result. See, e. g., Fiedler and Ptàk(1962), Horn and Johnson (1991), Murota and Sugihara (2022), Meyer (2000).

Theorem 6. For a square matrix $B \in Z$, B of order n, the following conditions are equivalent.

1. For any $c \geqq [0]$ there exists $x \geqq [0]$ such that $Bx = c$.
2. There exist $c > [0]$ and $x \geqq [0]$ such that $Bx = c$.
3. There exist $c > [0]$ and $x > [0]$ such that $Bx = c$.
4. $\Delta_k > 0$ for all $k = 1, ..., n$ ("Hawkins-Simon conditions").
5. There exist lower and upper triangular matrices L and U, with positive diagonals and non positive off-diagonal elements, such that $B = LU$.
6. B is nonsingular and $B^{-1} \geq [0]$.

Proof.

$1 \Longrightarrow 2$: obvious.

$2 \Longrightarrow 3$. Suppose that $By = d$ for some $y \geqq [0]$ and $d > [0]$. Let $\alpha > 0$ small enough that $d + \alpha(Bu) > [0]$, and let $c = d + \alpha(Bu) > [0]$ and $x = y + \alpha u > [0]$. Here

$$u = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Then we have $Bx = By + \alpha(Bu) = c$.

$3 \Longrightarrow 4$. By induction: the property holds for $n = 1$. Assume that $3 \Longrightarrow 4$ holds for $n-1$. We have

$$b_{11} = \left(c_1 - \sum_{j \neq 1} b_{1j} x_j \right) \diagup x_1 > 0$$

since $b_{1j} \leqq 0$, $\forall j \neq 1$.

Let be

$$D = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 0 \\ -\frac{b_{21}}{b_{11}} & 1 & \cdots & \cdots & 0 \\ -\frac{b_{31}}{b_{11}} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -\frac{b_{n1}}{b_{11}} & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then

$$DB = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & & & \\ \vdots & & \tilde{B} & \\ 0 & & & \end{pmatrix}$$

where $\tilde{b}_{ij} = b_{ij} - \frac{b_{i1}}{b_{11}} \leqq 0$, for all $i, j \geqq 2$, $i \neq j$.

Also $(Dc)_i = c_i - \frac{b_{i1}}{b_{11}} c_1 > 0$ for all $i \geqq 2$.

Letting $y = (x_2, ..., x_n)^\top > [0]$ and $d = ((Dc)_2, ..., (Dc)_n)^\top > [0]$, we have $\tilde{B}y = d$. Therefore, by the induction hypothesis, $\Delta_\ell(\tilde{B}) > 0$ for all $\ell = 1, ..., n-1$. Hence, for all $k = 1, ..., n$, we have $\Delta_k = b_{11}\Delta_{k-1}(\tilde{B}) > 0$.

$4 \Longrightarrow 5$. By induction: the property holds for $n = 1$. Assume that $4 \Longrightarrow 5$ holds for $n-1$. Suppose that B satisfies 4. Let D and $\tilde{B}$ be as in the previous proof. Since $\Delta_\ell(\tilde{B}) = \frac{1}{b_{11}} \Delta_{\ell+1}(B) > 0$ for all $\ell = 1, ..., n-1$, $\tilde{B}$ is written as

$$\tilde{B} = \tilde{L}\tilde{U}$$

by the induction hypothesis.

Let be

$$L = D^{-1} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{L} & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & \tilde{L} & \\ * & & & \end{pmatrix},$$

$$U = \begin{pmatrix} b_{11} & \cdots & \cdots & b_{1n} \\ 0 & & & \\ \vdots & & \tilde{U} & \\ 0 & & & \end{pmatrix}$$

where $\ell_{i1} = \frac{b_{i1}}{b_{11}} \leqq 0$ for $i = 2, ..., n$. Then $B = LU$.

5 $\Longrightarrow$ 6. L^{-1} and U^{-1} exist and are nonnegative. Therefore $B^{-1} = U^{-1}L^{-1} \geqq [0]$.

6 $\Longrightarrow$ 1.For any $c \geqq [0]$, $B^{-1}c \geqq [0]$. Let $x = B^{-1}c$. $\square$

Remark 6. The factorization appearing in 5. is called *LU (i. e. lower-upper) factorization* or *LU decomposition* and it was used perhaps for the first time in the theory of M-matrices, by Fiedler and Ptàk (1962). When the matrix is symmetric and definite positive, this factorization is also called "Cholesky factorization". See, e. g., Meyer (2000).

C) **The proof of Kurz and Salvadori (1995).**

These authors give a direct proof of the following equivalence, previously proved at point A), Theorem 5:

$$\rho > \lambda^*(A) \iff \text{All leading principal minors of } (\rho I - A) \text{ are positive,}$$

where $A \geqq [0]$, square of order n. Their proof, we present for the reader's convenience, is based on the fact that the above conditions are in turn equivalent to $(\rho I - A)^{-1} \geq [0]$. The said authors prove, in an autonomous way, the following useful results.

Lemma 2. Let $A \geqq [0]$ be of order n, and let $\lambda^*(A)$ be its Frobenius eigenvalue. Moreover, let be $\rho > \lambda^*(A)$, $\rho \in \mathbb{R}$, and $\upsilon = (1/\rho)$. Then:

(a) $\lim_{k \longrightarrow +\infty} (\upsilon A)^k = [0]$;

(b) $[I - \upsilon A]^{-1} = \lim_{k \longrightarrow +\infty} \sum_{i=0}^{k} (\upsilon A)^i$;

(c) $[\rho I - A]^{-1} = \frac{1}{\rho} \lim_{k \longrightarrow +\infty} \sum_{i=0}^{k} \left(\frac{1}{\rho} A\right)^i$.

Theorem 7 (Kurz and Salvadori). Let $A \geqq [0]$ be of order n and let $\lambda^*(A)$ be its Frobenius eigenvalue. Then in the matrix $[\rho I - A]$ it results $\rho > \lambda^*(A)$ if and only if all leading principal minors of $[\rho I - A]$ are positive.

Proof. Let $\Delta_1 = \rho - a_{11}$, $\Delta_2, ..., \Delta_n = \det(\rho I - A)$ be the leading principal minors of $[\rho I - A]$. In order to prove the "only if" part, let us recall the expression of the inverse by means of its adjoint matrix and the result (c) of Lemma 2. Then we have

$$\frac{\Delta_{n-1}}{\Delta_n} = (e^n)^\top [\rho I - A]^{-1} e^n \geqq \frac{1}{\rho} > 0$$

(e^n is the n-th unit vector of $\mathbb{R}^n : e^n = [0, 0, ..., 1]^\top$).

Then, consider the matrix B_n obtained from A by deleting the n-th row and the n-th column. Since it is still true that $\exists x \geqq [0] : x^\top [\rho I - B_n] > [0]$, then

$$\frac{\Delta_{n-2}}{\Delta_{n-1}} = (e^{n-1})^\top [\rho I - B_n]^{-1} e^{n-1} \geqq \frac{1}{\rho} > 0.$$

By iteration we obtain that all $\Delta' s$ have the same sign, and since $\Delta_1 > 0$, all of them are positive.

In order to prove the "if" part, let us remark that the theorem is true for $n = 1$. Then we prove that if the theorem is true for $n = s - 1$, it is true for $n = s$. In order to simplify the notation, let us put

$$[\rho I - B_{s+1}] = \begin{bmatrix} \rho I - B_s & -b^s \\ -(d^s)^\top & \rho - a_{ss} \end{bmatrix},$$

where the matrix B_i $(i = s, s+1)$ is obtained from A by deleting all rows and columns from the i-th to the n-th, the vector b^s is constituted by the first $s-1$ elements of the s-th column of A, and the vector d^s is constituted by the first $s-1$ elements of the s-th row of A. Since $\Delta_1, \Delta_2, ..., \Delta_{s-1}$ are positive, we obtain from the theorem that $[\rho I - B_s]$ is invertible and its inverse is semipositive. We observe that the following equality holds:

$$\begin{bmatrix} I & [0] \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ [0] & D - CA^{-1}B \end{bmatrix}$$

and hence

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B).$$

We have

$$\Delta_s = \Delta_{s-1}\left\{(\rho - a_{ss}) - (d^s)^\top [\rho I - B_s]^{-1} b^s\right\}$$

and since both Δ_s and Δ_{s-1} are positive, we obtain that

$$\beta := (\rho - a_{ss}) - (d^s)^\top [\rho I - B_s]^{-1} b^s > 0.$$

Hence we can choose a scalar $\epsilon > 0$ such that $\beta > \epsilon e^\top [\rho I - B_s]^{-1} b^s$. Let be

$$z^\top := (d^s)^\top [\rho I - B_s]^{-1} + \epsilon e^\top [\rho I - B_s]^{-1}$$

and

$$\alpha := (\rho - a_{ss}) - z^\top b^s.$$

Then $z \geq [0]$, $\alpha = \beta - \epsilon e^\top [\rho I - B_s]^{-1} b^s > 0$ and

$$\begin{bmatrix} z \\ 1 \end{bmatrix}^\top [\rho I - B_{s+1}] = \begin{bmatrix} z \\ 1 \end{bmatrix}^\top \begin{bmatrix} \rho I - B_s & -b^s \\ -(d^s)^\top & \rho - a_{ss} \end{bmatrix} = \begin{bmatrix} \epsilon e \\ \alpha \end{bmatrix}^\top > [0,]$$

which completes the proof. □

D) **The proof of Fujimoto (2007).**

Fujimoto (2007) obtains both the Hawkins-Simon conditions (i. e. the Georgescu-Roegen conditions, referred to the leading principal minors) and the original Hawkins-Simon conditions (i. e. referred to all principal minors) as quite simple corollaries of a result of Banachiewicz on the inversion of partitioned matrices. We have to note that the Polish mathematician Tadeusz Banachiewicz introduced in 1938 also the so-called LU-factorization of the previous point B). We only report the Banachiewicz identity, referring the reader to the paper of Fujimoto (2007) for the related proofs of the Hawkins-Simon conditions. Let be given

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

with A square and nonsingular, of order $n \geqq 2$, E square and nonsingular, of order p, $1 \leqq p < n$, H square of order $q = n-p$. Then the Banachiewicz identity is

$$A^{-1} = \begin{bmatrix} E^{-1} + E^{-1}FS^{-1}GE^{-1} & -E^{-1}FS^{-1} \\ -S^{-1}GE^{-1} & S^{-1} \end{bmatrix},$$

where S, assumed nonsingular, is the *Schur complement,* defined by

$$S = H - GE^{-1}F.$$

The proofs of Fujimoto are very short and elegant, even if they rely on the above result, not too known outside the specialists in Matrix Analysis.

E) **The proof of Morishima (1964).**

Morishima (1964) takes into consideration the classical input-output system

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + c_i, \quad i = 1, ..., n, \tag{3}$$

where $a_{ij} \geqq 0$ and $c_i \geqq 0$. This author then assumes that $A = [a_{ij}]$ is *indecomposable,* hence he obtains a strong result.

Theorem 8 (Morishima). A necessary and sufficient condition that x satisfying (3) be all positive for any $c \geq [0]$, is that all leading principal minors of $[I - A]$ be positive, i. e.

$$\Delta_1 > 0, \ \Delta_2 > 0, ..., \Delta_n > 0.$$

Note of Morishima: "A sufficient condition that a nonnegative, indecomposable matrix satisfy the Hawkins-Simon condition, is that all column sums of the matrix are not greater than one, and at least one of them is less than one. See Solow (1952)".

The proof of Morishima is quite simple, by induction, but avoiding to transform the system $(I - A)x = c$ into a triangular form, through a Gaussian elimination, as done, for example in Nikaido (1968, 1970), and in the previous point A). A proof quite similar to the proof of Morishima is considered by

Lippi(1979). Also Kato, Matsumoto and Sakai (1972) take into consideration system (3), rewritten in the form

$$\sum_{j=1}^{n} a_{ij}x_j = k_i, \ i = 1, ..., n,$$

where $a_{ij} \leqq 0, \ \forall i = j$. However, their proof, by induction, follows the same lines of the proofs of Nikaido (1968, 1970).

F) **The chain of proofs of Horn and Johnson (1991).**

Horn and Hohnson (1991) present a quite long list of properties equivalent to the Hawkins-Simon conditions (however their list is not so long as the one considered, e. g., by Berman and Plemmons (1994), Giorgi (2022), Magnani and Meriggi (1981), Plemmons (1977)). The most known characterizations of nonsingular M-matrices, considered by Horn and Johnson, are described in the following result. For these authors a Z-matrix B is an M-matrix if B is *positive stable,* that is each eigenvalue of $B \in Z$ has a positive real part.

Theorem 9. If the real square matrix B, of order n, is a Z-matrix, then the following statements are equivalent.

1) B is positive stable, that is B is an M-matrix.
2) $B = \alpha I - A$, $A \geqq [0]$, $\alpha > \lambda^*(A)$.
3) Every real eigenvalue of B is positive.
4) $B + tI$ is nonsingular for all $t \geqq 0$.
5) $B + D$ is nonsingular for every nonnegative diagonal matrix D.
6) All principal minors of B are positive.
7) The sum of all principal minors of B of order k, $k = 1, ..., n$, is positive.
8) The leading principal minors of B are positive.
9) $B = LU$, where L is lower triangular and U is upper triangular and all the diagonal entries of each are positive.
10) For each nonzero $x \in \mathbb{R}^n$, there is an index $1 \leqq i \leqq n$ such that $x_i(Bx)_i > 0$.
11) For each nonzero $x \in \mathbb{R}^n$, there is a positive diagonal matrix D such that $x^\top BDx > 0$.
12) There is a positive vector $x \in \mathbb{R}^n$, with $Bx > [0]$.
13) B is nonsingular and $B^{-1} \geq [0]$.
14) $Bx \geqq [0]$ implies $x \geqq [0]$.
15) There is a positive diagonal matrix D such that $DB + B^\top D$ is positive definite (This condition is also known as the "Lyapunov condition").

The authors do not give a complete proof of the equivalences of all the above properties, however they offer a useful sample of the various implications. This can indeed be useful, as they do not follow in general the approach which may be found in several books of mathematical economics. In general their proofs are quite short.

G) **The chain of proofs of Meyer (2000).**

Also Meyer (2000) presents a chain of proofs of conditions equivalent to the Hawkins-Simon conditions. These proofs may be interesting from a didactical point of view. For this author, M-matrices are real nonsingular matrices B of order n, such that $b_{ij} \leqq 0$, $\forall i \neq j$, and $B^{-1} \geq [0]$.

First, this author proves the following characterizations.

1. B is an M-matrix if and only if there exist a matrix $A \geqq [0]$ and a real number $\rho > \lambda^*(A)$ such that $B = \rho I - A$.

2. B is an M-matrix if and only if $\mathrm{Re}(\lambda) > 0$, for all λ such that $\det(B - \lambda I) = 0$.

3. B is an M-matrix if and only if all its principal minors are positive.

Subsequently, this author proves that the following statements are equivalent:

(a) B is an M-matrix.

(b) All leading principal minors of B are positive.

(c) B has an LU factorization, and both L and U are M-matrices.

(d) There exists a vector $x > [0]$ such that $Bx > [0]$.

(e) Each $b_{ii} > 0$ and BD is diagonally dominant (in the usual sense of Hadamard) for some diagonal matrix matrix D with positive diagonal entries.

(f) $Bx \geqq [0]$ implies that $x \geqq [0]$, i. e. B is *monotone.*

The author then proves the following implications (in the Solution Manual).

$(a) \implies (b)$: point 3 of the previous characterizations.

$(b) \implies (c)$: by induction on the size of B.

$(c) \implies (d)$: $B = LU$, with L and U M-matrices, implies $B^{-1} = U^{-1}L^{-1} \geq [0]$, so, if $x = B^{-1}e$, where $e = [1, 1, ..., 1]^\top$, then $x > [0]$ (otherwise B^{-1} would have a zero row, and B would be singular), and $Bx = e > [0]$.

$(d) \implies (e)$. If $x > [0]$ is such that $Bx > [0]$, define $D = diag(x_1, x_2, ..., x_n)$ and set $A = BD$, which is clearly a Z-matrix. For $e = [1, 1, ..., 1]^\top$, notice that $Ae = BDe = Bx > [0]$ says each row sum of $A = BD$ is positive. In other words, for each $i = 1, 2, ..., n$,

$$0 < \sum_j a_{ij} = \sum_{j \neq i} a_{ij} + a_{ii} \implies a_{ii} > \sum_{j \neq i} -a_{ij} = \sum_{j \neq i} |a_{ij}|$$

for each $i = 1, 2, ..., n$.

$(e) \implies (f)$. See Meyer (2000).

$(f) \implies (e)$. We first prove that if B is monotone, then B is nonsingular. From $Bx = [0]$ and $B(-x) = [0]$ we obtain $x \geqq [0]$ and $x \leqq [0]$, respectively, that is $x = [0]$. So the system $Bx = [0]$ has only the trivial solution $x = [0]$ and therefore B is nonsingular. On the other hand, let $(e^1, e^2, ..., e^n)$ be the canonical basis in $\mathbb{R}^n$. As $A(A^{-1}e^i) \geq [0]$, $i = 1, ..., n$, so $A^{-1} \geq [0]$. $\square$

Remark 7. A real square matrix A, of order n, not necessarily $A \in Z$, is said to be *monotone* if

$$Ax \geqq [0] \implies x \geqq [0] .$$

A classical result of Collatz (1952, 1966) states that A has a semipositive inverse if and only if A is monotone. The result of Collatz has been generalized to non necessarily square matrices by Mangasarian (1968).

Remark 8. There exist several proofs of the Hawkins-Simon conditions, based on economic considerations. See, e. g., Dasgupta (1984, 1992), Fujita (1991), Jeong (1982). These proofs are interesting from an economic point of view, but usually they are lacking in mathematical accuracy. The "alternative proof" of O'Neill and Wood (1999) is really a revisitation of the Gaussian elimination method, together with the induction method, used by Nikaido (1968, 1970). Curiously, these books of Nikaido are not quoted by these authors. In the same paper of O'Neill and Wood (1999), some computational aspects of the Hawkins-Simon conditions are discussed. D. Xie (1992) asserts to be "surprised how little the proofs (of the Hawkins-Simon conditions) make use of mathematical induction". Also this author does not quote Morishima (1964), Nikaido (1968, 1970), etc. Furthermore, Lemma 1 of the paper of Xie is clearly wrong, as it is asserted that a matrix H, of order $(n+1)$, $H \in Z$, is an M-matrix, i. e. it satisfies the Hawkins-Simon conditions, if and only if $h_{n+1,n+1} > 0$, which obviously is incorrect.

Remark 9. Bergthaller and Dragomirescu (1971) and Giorgi (1987) use a suitable theorem of the alternative for linear systems, in order to give another condition equivalent to the Hawkins-Simon conditions for a Z-matrix B, of order n. More precisely: the system $Bx > [0]$, $x \geqq [0]$, is consistent if and only if the system $Bx \leqq [0]$, $x \geq [0]$, is inconsistent. Furthermore, Giorgi (1987), following Magnani and Meriggi (1981), points out what previously said in Remark 1, which allows to obtain other characterizations of M-matrices from the usual characterizations appearing in the literature.

3. Extensions and Related Topics

(I) *The weak Hawkins-Simon conditions.*

Fujmoto and Ranade (2004), Bidard (2007) and Ranade and Fujimoto (2005 a), b)), have studied the *weak Hawkins-Simon conditions,* which are the usual Hawkins-Simon conditions, expressed in terms of positivity of the *leading principal minors,* but referred to a square matrix A, *not necessarily in the class of Z-matrices.* In particular, Fujimoto and Ranade (2004) prove that the weak Hawkins-Simon conditions are necessary for a real square matrix A to have a semipositive inverse, $A^{-1} \geq [0]$, after a suitable permutation of columns (or rows). Their proof is easy and uses the Gaussian elimination method. More precisely, these authors prove the following result.

Theorem 10. Let A be a real square matrix of order n and let $A^{-1} \geq [0]$. Then the weak Hawkins-Simon conditions are satisfied by A, i. e. there is a suitable permutation of columns (or rows) such that the resulting matrix satisfies the Hawkins-Simon conditions.

As a corollary, the usual necessary and sufficient Hawkins-Simon conditions for $A \in Z$ to have a semipositive inverse, are obtained. Also this corollary is proved in a rather short way.

Bidard (2007) has relaxed the assumptions of Theorem 10 on A^{-1}, as he proved that if the last column of the inverse of a real square matrix is strictly positive, the matrix enjoys of the property described by Theorem 10. Other sufficient conditions for the validity of the said property are contained in the papers of Ranade and Fujimoto (2005 a),b)).

(II) *The Hawkins-Simon conditions for linear joint production models.*

Applications of Hawkins-Simon conditions (or modified Hawkins-Simon conditions) to linear joint production models are somewhat problematic. We take into consideration the following Leontief-Sraffa-type joint production model:

$$Ax + c = Bx, \tag{4}$$

where $A \geq [0]$, $B \geq [0]$, are square of order n, Bx represents the *gross output vector,* Ax the *vector of inputs requirements,* and $c \geq [0]$ is a given final demand vector. Obviously, if $(B - A)^{-1} \geq [0]$ or even $(B - A)^{-1} > [0]$, there are no problems in solving system (4). We recall that a square matrix A of order n has a semipositive inverse if and only if A is *monotone* (Collatz (1966)):

$$Ax \geqq [0] \implies x \geqq [0].$$

However, the above condition is of little utility in applications, unless $A \in Z$. Following Schefold (1978), with reference to a Leontief-Sraffa model with joint production, described by the pair (A, B), a commodity is said to be *separately producible* if it is possible to produce a net output consisting of a unit of that commodity alone with a nonnegative intensity vector. That is, commodity j is separately producible if and only if there is a nonnegative vector x^j such that

$$(B - A)x^j = e^j$$

where e^j is the unit vector of $\mathbb{R}^n$. A technique (A, B) is called *all-productive* if all commodities are separately producible, that is for every nonnegative vector y there exists a nonnegative vector x such that

$$(B - A)x = y.$$

Peris and Villar (1993) say that system (4) is *strongly solvable* if the above condition holds. This condition is equivalent to $(B - A)^{-1} \geq [0]$. Hence, (A, B) is called by Schefold (1978) "all-productive" if $(B - A)^{-1} \geq [0]$. The technique (A, B) is called by Schefold (1978) "all-engaging" if $(B - A)^{-1} > [0]$. Schefold proves that all-productive systems and all-engaging systems have almost all properties of single production systems. This is true also for the Sraffa price system in joint production:

$$(1+r)p^{\top}A + w\ell^{\top} = p^{\top}B,$$

where $r \geqq 0$ is the profit rate, $w \geqq 0$ is the wage rate, $p \geq [0]$ is the vector of equilibrium prices and $\ell > [0]$ is the vector of direct labour requirements. If (A, B) is all-productive, the above system can be rewritten as

$$p^{\top}(B-A) = rp^{\top}A + w\ell^{\top},$$

that is

$$p^{\top} = rp^{\top}A(B-A)^{-1} + w\ell^{\top}(B-A)^{-1}.$$

Hence, all properties of single product techniques with respect to the price system hold, since the Perron-Frobenius theorem and the theory of M-matrices can be applied to the nonnegative matrix $A(B-A)^{-1} \equiv H$. Indeed, we have

$$p^{\top} - rp^{\top}A(B-A)^{-1} = w\ell^{\top}(B-A)^{-1},$$

that is

$$p^{\top}(I - rA(B-A)^{-1}) = w\ell^{\top}(B-A)^{-1},$$

that is

$$p^{\top} = w\ell^{\top}(B-A)^{-1}(I - rA(B-A)^{-1})^{-1},$$

and, if $r > 0$, we have

$$(I - rA(B-A)^{-1})^{-1} \geq [0]$$

if and only if $\frac{1}{r} > \lambda^*(H)$.

Given a real square matrix M of order n, any representation of M in the form

$$M = B - A,$$

where B and A are square of the same order n, is called a "split" or a "splitting" of M. There are some conditions assuring that $M^{-1} \geq [0]$ in terms of specific types of splittings. See, e. g., Berman and Plemmons (1994). Usually, these conditions require, besides other properties, $B^{-1} \geq [0]$. However, in a Leontief-Sraffa model with joint production, this is a drawback, as a result of Johnson (1983) states that a semipositive square matrix has a semipositive inverse only if it is a diagonal matrix or a permutation of a diagonal one. This implies that joint production is ruled out in this case. See further in the present paper for other considerations on splittings of a square matrix.

Obviously, if $(B-A) \in Z$, i. e. $b_{ij} \leqq a_{ij}$, $\forall i \neq j$, then we can apply to $(B-A)$ the Hawkins-Simon conditions (or other equivalent conditions) in order to obtain $(B-A)^{-1} \geq [0]$. The assumption $(B-A) \in Z$ is made, for instance, by Fujimoto (1979) and by Bapat, Olesky and Van Den Driessche (1995). However, as admitted by Fujimoto (1979), this assumption says that there exists no net joint production; see also the comments of Peris and Villar (1993). Other assumptions on the pair (A, B), useful to extend the classical Perron-Frobenius theorem and to study linear joint production models, are made by Mangasarian (1971). In the economic literature this approach was anticipated by Hicks (1965). See also the papers of Giorgi and Magnani (1978), Łos (1971), Punzo (1980), Fujimoto and Krause (1988), Mehrmann, Olesky, Phan and Van Den Driessche (1999).

We consider the following version of the Mangasarian condition:

$$Bx \geqq [0] \implies Ax \geqq [0] .$$

This property is called by Fujimoto and Krause (1988) "non dominance property": it simply says that no positive output is possible without consuming some inputs, which is obviously acceptable in all production models. It can be proved that the non dominance property is equivalent to:

The cone $\left\{y^\top B,\ y \geqq [0]\right\}$ includes the cone $\left\{y^\top A,\ y \geqq [0]\right\}$.

Furthermore, Mangasarian (1971) has proved that the said condition is also equivalent to the existence of a square matrix $X \geqq [0]$ of order n, such that

$$A = XB. \tag{5}$$

It turns out, when B is nonsingular, that (5) is equivalent to

$$AB^{-1} \geqq [0] .$$

The condition of Mangasarian is very useful in obtaining generalizations of the Perron-Frobenius theorem for the pair (A, b) and it has been independently used by Łos (1971) in the study of the existence of solutions of the classical von Neumann economic model of balanced growth (see also the interesting considerations of Abraham-Frois and Berrebi (1976)). However, the Mangasarian condition alone is not sufficient to get $(B - A)^{-1} \geq [0]$. We have to add some other properties, possibly avoiding to impose $B^{-1} \geq [0]$ and/or $A^{-1} \geq [0]$. A nice result of Peris (1991) answers this question. A splitting of a square matrix M, of order n, is called a *positive splitting* if $M = B - A$, with $A \geqq [0]$ and $B \geqq [0]$.

Definition 3. A positive splitting $M = B - A$ of a square matrix M of order n is called a *B-splitting* if B is nonsingular and

(a) $Bx \geqq [0] \implies Ax \geqq [0]$.

(b) For all $x \in \mathbb{R}^n$:

$$\begin{pmatrix} M \\ B \end{pmatrix} x \geqq [0] \implies x \geqq [0] .$$

Note that any Z-matrix admits a B-splitting, however the converse is not true (Peris (1991) gives a counterexample). For those matrices admitting a B-splitting the following result holds.

Theorem 11. Let M be a square matrix of order n, such that $M = B - A$ is a B-splitting. Then the following conditions are equivalent:

(a) $M^{-1} \geq [0]$.

(b) $\lambda^*(AB^{-1}) < 1$.

(c) There exists $x \geq [0]$ such that $Mx > [0]$.

(d) The matrix $(I - AB^{-1})$ satisfies the Hawkins-Simon conditions.

The following result, always due to Peris (1991), characterizes square matrices with a semipositive inverse in terms of B-splittings.

Theorem 12. For a square matrix M of order n, the following conditions are equivalent:

(a) $M^{-1} \geq [0]$.

(b) M admits a B-splitting $M = B - A$, such that $\lambda^*(AB^{-1}) < 1$.

(III) *Metzler matrices.*

Metzler matrices or *Metzlerian matrices* are those square matrices A of order n, such that $-A \in Z$, i. e. such that $a_{ij} \geqq 0$, $\forall i \neq j$. These matrices play a fundamental role in the study of stability of equilibrium solutions of Walrasian competitive markets. See, e. g., Kemp and Kimura (1978), Murata (1977), Takayama (1985), Woods (1978). On the grounds of what expounded in Section 2, it is quite easy to prove the following results.

Theorem 13. Let A be a Metzler matrix of order n. Then the following conditions are equivalent.

(1) $A = M - \alpha I$, where $M \geqq [0]$ and $\alpha > \lambda^*(M)$.

(2) A is a *stable matrix,* i. e. $\text{Re}(\lambda) < 0$, where λ is any eigenvalue of A.

(3) A is *Hicksian stable,* i. e. every odd order principal minor of A is negative and every even order principal minor of A is positive.

(4) The *leading* principal minors of A obey the rule: the sign of the i-th leading principal minor is $(-1)^i$, $i = 1, ..., n$.

(5) A^{-1} exists and $A^{-1} \leq [0]$.

(6) There exists a vector $x \geqq [0]$ such that $Ax < [0]$.

(7) For any $c \leqq [0]$, there exists an $x \geqq [0]$ such that $Ax = c$.

(8) A has a quasidominant diagonal, with all diagonal elements negative.

(9) There exists a lower triangular matrix T_0 and an upper triangular matrix T_1, such that T_0 and T_1 are Metzler matrices, T_0 and T_1 are Hicksian stable matrices, and $A = T_0 T_1$.

Moreover, it can be proved (see Kemp and Kimura (1978)) that the eigenvalue of a Metzler matrix with largest real part is real and has an associated semipositive vector, that is Metzler matrices enjoy the main property of the Perron-Frobenius theorem for nonnegative square matrices.

(IV) *Nonlinear generalizations of the Leontief models.*

Several authors have been concerned with nonlinear generalizations of the classical Leontief models. See, e. g., Chander (1983), Chien and Chan (1979), Fujimoto, Herrero and Villar (1985), Fujimoto, Silva Reus and Villar (2003), Iritani (1990), Lahiri (1976, 1985), Lahiri and Pyatt (1980), Sandberg (1973), Silva Reus (1986). The various solvability conditions studied by the said authors may be considered as a generalization to the nonlinear case of the classical Hawkins-Simon conditions. In particular, the paper of Lahiri (1985) is concerned with some comparisons between various "nonlinear Hawkins-Simon conditions" proposed in the literature.

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