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Product Design in a Cournot Duopoly*

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Abstract

Product design is studied in a simple duopoly where firms compete à la Cournot, goods are hedonically differentiated and consumers have preferences defined over characteristics. What we find is that, in equilibrium, firms choose the same product's design. This results in *horizontal* product differentiation being minimal.

Keywords: Hedonic Product Differentiation, Horizontal Differentiation, Cournot Competition.

JEL Classification: D43, L13, L20.

1 Introduction

Since the seminal work by Hotelling (1929), firms' deifferentiation choices have predominantly been studied by means of the baseline Hotellian framework. Hotelling (1929)'s conclusion is that firms' tendency is to concentrate at the center of the market[1]. Two are the key elements defining the Hotellian framework: i) competition is in prices, and ii) consumers have unit demand and are assumed to buy just one of the available goods.

The literature on firms' product differentiation choices contains few attempts trying to embody quantity, instead of price, competition in the typical Hotelling (1929)'s framework. Among these, the closest, in logic, to our work is Hamilton, et. al (1994). They

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[1]This is known as *principle of minimal differentiation.* It was later showed, by D'Aspremont et. al (1979), that a pure strategy Nash equilibrium yielding minimal differentiation can never occur in the original Hotelling (1929)'s framework with linear transportation costs. However, they showed that for quadratic transportation costs existence is restored but firms will chose to maximize differentiation, i.e to locate at the two extremes of the market line.

explicitly study quantity competition in the typical spatial setting as originally proposed by Hotelling. Their conclusion is, somehow surprisingly, that firms will never chose to locate exactly in the same spot. Thus, the principle of minimal product differentiation is seen not to hold. The reason for this is to be found, essentially, in the fact that they derive their inverse demands starting from market areas segmentation based on firms' prices. It follows that inverse demands suffer the consequences of the discontinuities that typically arise when segmenting the market in various market areas[2]. Because of this, they are not able to provide a close-form solution for the model, but rely on numerical procedures in order to define equilibrium solutions.

In the present work, we analyze a duopoly where firms choose the characteristics of their products in a Lancasterian setting, namely where the consumers' utility function is defined over the vectors of characteristics rather than over goods; that is, a good is defined as a vector in the space of characteristics. In particular, we will consider the combinable consumption version of the Lancasterian setting[3]. This refers to the possibility, for consumers, to combine different characteristics together to reach a certain level of utility. While in the typical product differentiation framework, inherited from Hotelling (1929), consumers consume "products", here they consume the characteristics they can combine through their purchase of products: consumers display love for variety[4], in the sense that they do not have unit demand but purchase a bundle of goods.

This way of modeling product differentiation has never been sufficiently considered[5]. However, recently, the work by Pellegrino (2023) showed how the combinable consumption version of the Lancasterian setting can be adopted as a foundation for a computable oligopoly model, to capture key consequences of market power. We assume that firms chose their product design, and than they compete in quantities.

We will consider a representative consumer whose preferences are described by a quasi-linear additively-separable utility function. This particular utility structure, when considering hedonically differentiated goods, naturally generates the workhorse framework introduced by Singh and Vives (1984). This feature provides a way for key utility parameters (in particular the parameter defining the degree of substitutability between goods) to be endogenized as they result from the firms' differentiation choices[6]. The model is essentially a two-stage game where in stage one firms decide on their respective good's design, and in stage two they compete by setting their quantities. We will show that a subgame perfect Nash equilibrium exists and results in both firms choosing the same product's design. Product differentiation will therefore be minimal in the sense that the market provides a single variety. While, however, in Hotelling (1929) this is not a pure strategy Nash equilibrium (see d'Aspremont et. al (1979) and Osborne and Ptchick (1987)), here it is the unique Subgame Perfect Nash equilibrium of the game.

To check the robustness of this result we solve the model expanded with the inclusion

[2]On this see Gabszewicz, et. al (1986).

[3]See Lancaster (1966), (1971). For further discussions on the characteristics model see Friedman (1983).

[4]This concept was introduced by Dixit and Stiglitz (1977) and refers to the marginal utility of a new variety, at zero, being infinite. In the current setting it refers to the fact that consumers do not seek their most preferred variety, but instead purchase a bundle of goods.

[5]Models where the *non-combinable consumption* version of the Lancasterian setting is considered abound in the IO literature. See for example Irmen and Thisse (1998).

[6]In Singh and Vives (1984) the differentiation structure is exogenous.

of two typical costs structures: linear and quadratic. We find that, with a linear costs structure, the magnitude of the constant marginal cost impacts firms' design choices; while, with a quadratic costs structure, since only average cost impacts firms' design choices, the latter remain unchanged.

Section 2 exposes the details of the model. Section 3 contains our main, yet modest, result. Finally, section 4 concludes.

2 The model

Consider a duopolistic economy where goods are hedonically differentiated, i.e defined as vectors in the two-dimensional characteristics space $\mathbb{R}^2_+$, $s = \{1, 2\}$ denoting the two technologically available characteristics that can be embodied in different goods. The market is populated by two technologically identical firms which are denoted by $j \in \{1, 2\}$. Each firm sells *only one* differentiated good. This good is defined as a vector $\mathbf{a}_j \in \mathbb{R}^S_+$. Define the 2×2 matrix A to be the matrix composed by the vectors $\mathbf{a}_j$, $j \in \{1, 2\}$. We assume that $\sum_{s=1}^{2} a_{js}^2 = 1$ for any $j \in \{1, 2\}$. This *normalization* provides us a way to define the *degree of product differentiation*, or *similarity*, $\gamma = \mathbf{a}_j \cdot \mathbf{a}_k$. This defines the angle between the two vectors. Hence, it is a measure of relative product differentiation[7].

Given the semi-positiveness[8] of $\mathbf{a}_j$, $\gamma \in [0, 1]$. In particular, $\gamma = 0$ defines independent goods, $\gamma = 1$ perfect substitutes, and for $\gamma \in (0, 1)$ goods result imperfect substitutes. Note how the current structure does not consider the case of complementary goods. Anyway, it is important to see how in the current setting the so called *degree of substitutability* is directly determined by goods' similarity. Thus, when firms decide on their good's design, they are directly affecting the degree of substitutability. This is an important aspect of this modeling strategy since usually, in the literature on product differentiation[9], the degree of substitutability is taken as being *fully* exogeneous. The figure below displays an example of products' configuration.

The demand side of the economy is populated by a representative consumer having preferences defined over the space of characteristics $\mathbb{R}^2_+$. In order to acquire characteristics, the representative consumer decides how much of each good to consume. That is, she has as decision variable the vector of quantities $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2_+$. Given the market matrix A and the vector of quantities $\mathbf{q}$, the total amount of characteristics acquirable by the representative consumer is

$$\mathbf{x} = A\mathbf{q} \tag{1}$$

[7]This "circular" normalization is adopted because of its simplifying power. In particular, one could also chose a "linear" normalization such as $a_{j1} + a_{j2} = 1$. However, with the "linear" normalization, the definition of the *similarity* becomes more cumbersome. In general, the angle between two vectors is defined implicitly by $\cos \theta_{jk} = \frac{\mathbf{a}_j \cdot \mathbf{a}_k}{||\mathbf{a}_j|| \cdot ||\mathbf{a}_k||}$. Thus, when using the "linear" normalization, an additional term given by the product of the norms must be considered. This substantially increases the complexity of the model.

[8]A vector $\mathbf{a}_j \in \mathbb{R}^S$ is semi-positive if it is non-negative but not zero. In other words, it has at least one component which is strictly positive.

$$\mathbf{a}_j > 0 \quad \Longleftrightarrow \quad \mathbf{a}_j \in \mathbb{R}^S_+, \quad \mathbf{a}_j \neq \mathbf{0}.$$

[9]For example Singh and Vives (1984).

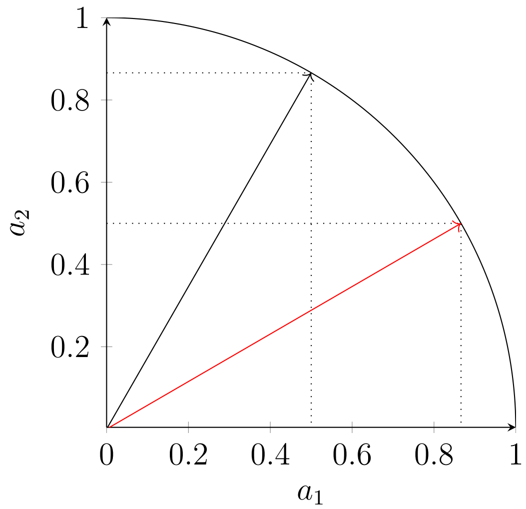

Figure 1: Graphical representation of a possible products' configuration. The black arrow corresponds to firm 1's product, $(a_{11}, a_{12}) = (1/2, \sqrt{3/4})$, and the red arrow to firm 2' product, $(a_{21}, a_{22}) = (\sqrt{3/4}, 1/2)$. Note that the configuration, generally, need not be symmetric.

where $x_s = \sum_j a_{js} q_j$ for any $s = \{1, 2\}$. The preferences of the representative consumer are described by the following utility function

$$U(x_0, \mathbf{x}) = x_0 + \sum_s \left(x_s - \frac{1}{2} x_s^2 \right) \tag{2}$$

where x_0 is an homogeneous good taken to be the numerier of the model. The utility function defined in (2), generates the well-known Singh and Vives (1984)'s specification. Thus, our characteristics representation is seen to be the *reasonable* foundation of the quadratic utility setting. In particular, define $\alpha_j = a_{j1} + a_{j2}$ for $j \in \{1, 2\}$, with $\alpha_j \in [1, \sqrt{2}]$[10]. We can write (2), expliciting the products' similarity γ, as

$$U(x_0, \mathbf{q}) = x_0 + \alpha_1 q_1 + \alpha_2 q_2 - \frac{1}{2}(q_1^2 + 2\gamma q_1 q_2 + q_2^2) \tag{3}$$

The representative consumer maximizes (3) with respect to the following standard budget constraint

$$B(\mathbf{q}, w_0) = \{(q_1, q_2) \in \mathbb{R}_+^2 \mid x_0 + p_1 q_1 + p_2 q_2 \leq w_0\} \tag{4}$$

where $w_0 > 0$ is the consumer's initial wealth (income). The first order conditions, assuming interior solutions, read

$$p_1 = \alpha_1 - q_1 - \gamma q_2$$
$$p_2 = \alpha_2 - \gamma q_1 - q_2$$

The above defines the inverse demand system. In linear algebra notation, it can be equivalently be expressed as

$$\mathbf{p}(\mathbf{q}) = \boldsymbol{\alpha} - \Gamma \mathbf{q} \tag{5}$$

[10]This range is readily derived from the "circular" constraint $a_{j1}^2 + a_{j2}^2 = 1$, $j \in \{1, 2\}$.

where $\Gamma = (A'A)$ is the matrix containing the products' similarities, and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. The indirect demand system in (5) is well defined in the region of the quantity space for which prices are positive.

The magnitude of the products' similarity γ is seen to directly affects the shape of the demand system, and hence of the competitiveness of the market. Indeed, the cases of $\gamma = 0$ and $\gamma = 1$ define very peculiar situations. In particular, $\gamma = 0$, i.e *independent goods* in conducive of *local monopoly power*. That is, firms, by focusing on providing a good embodying only one of the available characteristics are *de facto* monopolists in the market of the chosen characteristic. On the other hand, for $\gamma = 1$, i.e perfect substitutes, firms face fiercer competition since both provide an homogeneous good. Instead, whenever $\gamma \in (0, 1)$, and goods result imperfect substitutes, the outcome of the quantity competition depends on the intensity of γ.

When competing à la Cournot, firms take the inverse demand system, (5), as given. The profit function for firm $j \in \{1, 2\}$ are given, respectively, by $\pi_1(\mathbf{q}) = (\alpha_1 - q_1 - \gamma q_2)q_1$, and $\pi_2(\mathbf{q}) = (\alpha_2 - q_2 - \gamma q_1)q_2$. For given $\gamma \in [0, 1]$, straightforward computations lead to the following Cournot-Nash equilibrium.

$$q_1^C = \frac{1}{4-\gamma^2}\Big(2\alpha_1 - \gamma\alpha_2\Big) \tag{6}$$

$$q_2^C = \frac{1}{4-\gamma^2}\Big(2\alpha_2 - \gamma\alpha_1\Big) \tag{7}$$

Some simple algebra using (6) and (7), yields the following equilibrium profit functions.

$$\pi_1(\mathbf{q}^C) = \left(\frac{2\alpha_1 - \gamma\alpha_2}{4-\gamma^2}\right)^2 \tag{8}$$

$$\pi_2(\mathbf{q}^C) = \left(\frac{2\alpha_2 - \gamma\alpha_1}{4-\gamma^2}\right)^2 \tag{9}$$

Note that (8) and (9) are the same profit functions arising in the original Singh and Vives (1984) model. However, and it is important to stress this aspect once again, our modeling strategy permits us to endogenize the key differentiation parameters, α_1, α_2 and γ. In the following section we present our main result.

3 Products' design equilibrium

In this section we investigate the subgame perfect Nash equilibrium (SPNE) of the model, corresponding to the products' design selection game (stage one). Recall that for every firm a feasible differentiated good, $\mathbf{a}_j$, must be such that $a_{j1}^2 + a_{j2}^2 = 1$. Further, $\gamma = \mathbf{a}_1 \cdot \mathbf{a}_2$, and $\alpha_j = a_{j1} + a_{j2}$ for $j \in \{1, 2\}$. By combining these identities we can write γ, α_1 and α_2 in terms of just one of the components of the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Without loss of generality we fix the first component of the two vectors, a_{11} and a_{21}, to be the strategic variable of, respectively, firm 1 and firm 2. We have the following identities $\gamma = a_{11}a_{21} + \sqrt{(1 - a_{11}^2)(1 - a_{21}^2)}$, $\alpha_1 = a_{11} + \sqrt{1 - a_{11}^2}$, and $\alpha_2 = a_{21} + \sqrt{1 - a_{21}^2}$, with $a_{j1} \in [0, 1]$ for $j \in \{1, 2\}$. Before we proceed in exposing the main result of this work, it is worth noting that, from the previous identities, we are considering asymmetric "positions" on the unitary interval $[0,1]$. Nevertheless, this asymmetry in feasible "positions" does not prevent for a symmetric solution in the product space to arise in equilibrium[11]. In the Appendix we provide evidence that, for symmetric "positions" $a_{11}+a_{21}=1$, no pure strategy SPNE exists.

With the above identities, the profit functions (8) and (9), can be stated in terms of just the two strategic variables (a_{11},a_{21}).

$$\pi_1(a_{11},a_{21}) = \left[\frac{2a_{11}+2\sqrt{1-a_{11}^2}}{4-(a_{11}a_{21}+\sqrt{(1-a_{11}^2)(1-a_{21}^2)})^2} - \frac{(a_{11}a_{21}+\sqrt{(1-a_{11}^2)(1-a_{21}^2)})(a_{21}+\sqrt{1-a_{21}^2})}{4-(a_{11}a_{21}+\sqrt{(1-a_{11}^2)(1-a_{21}^2)})^2}\right]^2 \tag{10}$$

$$\pi_2(a_{11},a_{21}) = \left[\frac{2a_{21}+2\sqrt{1-a_{21}^2}}{4-(a_{11}a_{21}+\sqrt{(1-a_{11}^2)(1-a_{21}^2)})^2} - \frac{(a_{11}a_{21}+\sqrt{(1-a_{11}^2)(1-a_{21}^2)})(a_{11}+\sqrt{1-a_{11}^2})}{4-(a_{11}a_{21}+\sqrt{(1-a_{11}^2)(1-a_{21}^2)})^2}\right]^2 \tag{11}$$

(10) and (11) are easily seen to be continuous in the two strategic variables $(a_{11},a_{21})\in[0,1]^2$. Regarding differentiability, things are much more complex since even the analytical derivation of partial derivatives turns out to be particularly cumbersome. Anyway, plotting either of the two reveals that in $[0,1]^2$ we can expect partial derivatives to exist and to be well-behaved.

As it results evident by inspecting the above functions, the analytical derivation of firms' best replies is seen to be, in this setting, a particularly hard task. However, we can proceed in a more intuitive way, which will also provide us uniqueness of the equilibrium. The proposition below states our main result.

Proposition 1. *The unique subgame perfect Nash equilibrium of the products' design game is given by* $(a_{11}^*,a_{21}^*)=(1/\sqrt{2},1/\sqrt{2})$.

Proof. Consider the profit function of firm $j=1$, (10). By rewriting the nominator, and rearranging terms, define $A(a_{11},a_{21})=2a_{11}+2\sqrt{1-a_{11}^2}-a_{11}a_{21}(a_{21}+\sqrt{1-a_{21}^2})$ and $B(a_{11},a_{21})=(a_{21}\sqrt{1-a_{21}^2}+1-a_{21}^2)\sqrt{1-a_{11}^2}$. Denote the denominator by $D(a_{11},a_{21})$. Taking the logarithmic transformation of $\pi_1(a_{11},a_{21})$ yields

$$\ln[\pi_1(a_{11},a_{21})] = 2\ln[A(a_{11},a_{21})-B(a_{11},a_{21})] - 2\ln[D(a_{11},a_{21})] \tag{12}$$

From (12), we get that $\partial\ln[\pi_1(a_{11},a_{21})]/\partial a_{11}\geq 0$ in $(a_{11},a_{21})\in[0,1/\sqrt{2}]\times[0,1]$. That is, the profit function is increasing in a_{11} for $(a_{11},a_{21})\in[0,1/\sqrt{2}]\times[0,1]$. Proceeding in the same way, but considering the complement interval $[1/\sqrt{2},1]\times[0,1]$, shows that the profit function is decreasing in a_{11} for $(a_{11},a_{21})\in[1/\sqrt{2},1]\times[0,1]$. Thus, $a_{11}=1/\sqrt{2}$ is the only maximizer of $\pi_1(a_{11},a_{21})$ in $(a_{11},a_{21})\in[0,1]^2$. Since the two profit functions

[11] Recall that we defined the product space as the segment of the unitary circumference lying in the positive orthant of $\mathbb{R}^2$.

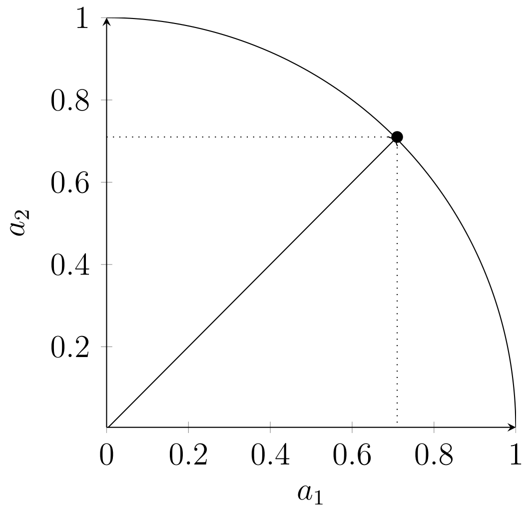

Figure 2: Graphical representation of the equilibrium outcome $(a_{11}^*, a_{21}^*) = (1/\sqrt{2}, 1/\sqrt{2})$. Both firms select the "central" design.

are symmetrical one another, firm $j = 2$'s profit function displays the same behaviour over the interval $[0, 1]^2$. From this we conclude that the only equilibrium is given by $(a_{11}^*, a_{21}^*) = (1/\sqrt{2}, 1/\sqrt{2})$. □

Proposition 1 establishes an interesting result. Namely, the strategic choice of goods' design yields *concentration* around a common design. The consequence of this, is that the market will provide a single homogeneous good (see Figure 2). The subsequent quantity competition will thus be the standard one arising when firms provide a single homogeneous good. In particular, since the chosen common design is the one containing the maximum amount of characteristics, the Cournot-Nash equilibrium outcome is $q_j^C = \sqrt{2}/3$ and $\pi_j(\mathbf{q}^C) = 2/9$, for $j \in \{1, 2\}$[12]. Denoting the standard duopoly Cournot competition equilibrium as $(\tilde{q}_1, \tilde{q}_2)$, and the resulting profit for the j-th firm by $\tilde{\pi}_j$, $j \in \{1, 2\}$, reveals that $\pi_j(\mathbf{q}^C) > \tilde{\pi}_j(\tilde{\mathbf{q}})$ for every $j \in \{1, 2\}$. That is, when firms can design their product over a multidimensional space, even if the tendency is towards *concentration*, the resulting profits are higher than in the standard case.

To check the robustness of this result we will consider two typical cost structures; linear: cq_j, $j \in \{1, 2\}$ with $c > 0$ denoting the common constant marginal cost, and quadratic: cq_j^2, $j \in \{1, 2\}$ with $c > 0$ denoting the common constant average cost. Consider first the linear cost structure. Profit function are given, respectively, by $\pi_1(\mathbf{q}) = (\alpha_1 - c)q_1 - q_1^2 - \gamma q_1 q_2$ and $\pi_2(\mathbf{q}) = (\alpha_2 - c)q_2 - q_2^2 - \gamma q_1 q_2$, with $c < \alpha_j$ for $j \in \{1, 2\}$. Since $\alpha_j \in [1, \sqrt{2}]$, $c < \alpha_j$ implies that $c \in [0, 1)$. For given products' designs, standard

[12]Note that the standard Cournot outcome, $q_j^C = 1/3$, arises in the special situation where both firms select a design containing only one of the two available characteristics. For example, when $a_{11} = a_{21} = 1$.

computations yield the following Cournot-Nash equilibrium

$$q_1^C = \frac{1}{4-\gamma^2}\Big[2\alpha_1 - \gamma\alpha_2 - c(2-\gamma)\Big] \tag{13}$$

$$q_2^C = \frac{1}{4-\gamma^2}\Big[2\alpha_2 - \gamma\alpha_2 - c(2-\gamma)\Big] \tag{14}$$

Equilibrium profits are easily derived and reads

$$\pi_1(\mathbf{q}^C) = \left[\frac{2\alpha_1 - \gamma\alpha_2 - c(2-\gamma)}{4-\gamma^2}\right]^2 \tag{15}$$

$$\pi_2(\mathbf{q}^C) = \left[\frac{2\alpha_2 - \gamma\alpha_2 - c(2-\gamma)}{4-\gamma^2}\right]^2 \tag{16}$$

We can now proceed in finding the SPNE of the game. By using the identities $\alpha_j = a_{j1} + \sqrt{1-a_{j1}^2}$ and $\gamma = a_{j1}a_{k1} + \sqrt{(1-a_{j1}^2)(1-a_{k1}^2)}$ for $j \neq k$ and $j \in \{1,2\}$ we can write (15) and (16) in terms of the two strategic variables a_{11}, a_{21}.

$$\pi_1(a_{11}, a_{21}) = \left[\frac{2a_{11} + 2\sqrt{1-a_{11}^2} - (a_{11}a_{21} + \sqrt{(1-a_{11}^2)(1-a_{21}^2)})(a_{21} + \sqrt{1-a_{21}^2})}{4 - (a_{11}a_{21} + \sqrt{(1-a_{11}^2)(1-a_{21}^2)})^2} - \frac{c(2 - a_{11}a_{21} - \sqrt{(1-a_{11}^2)(1-a_{21}^2)})}{4 - (a_{11}a_{21} + \sqrt{(1-a_{11}^2)(1-a_{21}^2)})^2}\right]^2 \tag{17}$$

$$\pi_2(a_{11}, a_{21}) = \left[\frac{2a_{21} + 2\sqrt{1-a_{21}^2} - (a_{11}a_{21} + \sqrt{(1-a_{11}^2)(1-a_{21}^2)})(a_{11} + \sqrt{1-a_{11}^2})}{4 - (a_{11}a_{21} + \sqrt{(1-a_{11}^2)(1-a_{21}^2)})^2} - \frac{c(2 - a_{11}a_{21} - \sqrt{(1-a_{11}^2)(1-a_{21}^2)})}{4 - (a_{11}a_{21} + \sqrt{(1-a_{11}^2)(1-a_{21}^2)})^2}\right]^2 \tag{18}$$

By a similar argument as the one used to prove Proposition 1, it is possible to show that, provided $c \leq 18/25$, the *unique* SPNE of the products' designs subgame is again given $(a_{11}^*, a_{21}^*) = (1/\sqrt{2}, 1/\sqrt{2})$. Essentially, for $c \leq 18/25$ the monotonicity of the log-transformed profit functions is preserved. Hence, the sign of the partial derivatives is unambiguous and leads to a clear definition of the SPNE. Conversely, for $18/25 < c < 1$, log-transformed profit functions become non-monotonic thus inducing discontinuities in the partial derivatives. Furthermore, the sign of the partial derivatives becomes ambiguous thus preventing us from defining the SPNE. From this, we conclude that, provided marginal cost to be sufficiently small, firms' tendency is still to select the same product's design; whether, if marginal cost is too high, no clear conclusion can be derived regarding firms' behaviour.

Consider now the quadratic cost structure cq_j^2 for $j \in \{1,2\}$, and $c > 0$. Profit functions are given, respectively, by $\pi_1(\mathbf{q}) = \alpha_1 q_1 - (1+c)q_1^2 - \gamma q_1 q_2$ and $\pi_2(\mathbf{q}) = \alpha_2 q_2 - (1+c)q_2^2 - \gamma q_1 q_2$. For given products' designs, the Cournot-Nash equilibrium can easily be derived and is given by

$$q_1^C = \frac{2\alpha_1(1+c) - \gamma\alpha_2}{4(1+c)^2 - \gamma^2} \tag{19}$$

$$q_2^C = \frac{2\alpha_2(1+c) - \gamma\alpha_1}{4(1+c)^2 - \gamma^2} \tag{20}$$

With (19)and (20) equilibrium profit functions can be derived. These are given by

$$\pi_1(\mathbf{q}^C) = (1+c)\left[\frac{2\alpha_1(1+c) - \gamma\alpha_2}{4(1+c)^2 - \gamma^2}\right]^2 \tag{21}$$

$$\pi_2(\mathbf{q}^C) = (1+c)\left[\frac{2\alpha_2(1+c) - \gamma\alpha_1}{4(1+c)^2 - \gamma^2}\right]^2 \tag{22}$$

Substituting the identities $\alpha_j = a_{j1} + \sqrt{1 - a_{j1}^2}$ and $\gamma = a_{j1}a_{k1} + \sqrt{(1 - a_{j1}^1)(1 - a_{k1}^2)}$, $j \neq k$ and $j \in \{1, 2\}$, yields (21) and (22) as functions of the two strategic variables (a_{11}, a_{21}).

$$\pi_1(a_{11}, a_{21}) = (1+c)\left[\frac{2(1+c)(a_{11} + 2\sqrt{1 - a_{11}^2})}{4(1+c)^2 - (a_{11}a_{21} + \sqrt{(1 - a_{11}^2)(1 - a_{21}^2)})^2} - \frac{(a_{11}a_{21} + \sqrt{(1 - a_{11}^2)(1 - a_{21}^2)})(a_{21} + \sqrt{1 - a_{21}^2})}{4(1+c)^2 - (a_{11}a_{21} + \sqrt{(1 - a_{11}^2)(1 - a_{21}^2)})^2}\right]^2 \tag{23}$$

$$\pi_1(a_{11}, a_{21}) = (1+c)\left[\frac{2(1+c)(a_{21} + 2\sqrt{1 - a_{21}^2})}{4(1+c)^2 - (a_{11}a_{21} + \sqrt{(1 - a_{11}^2)(1 - a_{21}^2)})^2} - \frac{(a_{11}a_{21} + \sqrt{(1 - a_{11}^2)(1 - a_{21}^2)})(a_{11} + \sqrt{1 - a_{11}^2})}{4(1+c)^2 - (a_{11}a_{21} + \sqrt{(1 - a_{11}^2)(1 - a_{21}^2)})^2}\right]^2 \tag{24}$$

An argument along the lines the one used to establish Proposition 1 reveals that with a quadratic cost structure, for any $c > 0$, the *unique* SPNE of the products' design subgame is again given by $(a_{11}^*, a_{21}^*) = (1/\sqrt{2}, 1/\sqrt{2})$. The economic intuition behind this result is particularly simple; namely, when marginal cost depends on firms' output, its impact on firms' product design choices vanishes. This is so because essentially the impact of marginal cost can be mitigated by reducing output, without impacting the subsequent products' designs competition. To see this, consider c_l and c_h with $c_l < c_h$; that is low and a high average costs. Denote the Cournot-Nash equilibrium when c_l as $\mathbf{q}_l^C$, and the Cournot-Nash equilibrium when c_h as $\mathbf{q}_h^C$. Clearly for any given combination of designs, $\mathbf{q}_l^C > \mathbf{q}_h^C$. However, the monotonicity of log-transformed profit functions (21) and (22) is unaffected by the magnitude of the average cost. The average cost increases both the nominator and the denominator of the profit functions, thus leaving the monotonicity of their log-transformations unchanged. For, the SPNE will be the same under either c_l or c_h.

4 Conclusion

We proposed a workable setting for studying product differentiation and firms' strategic design choice. The Lancasterian characteristics setting, in its *combinable consumption* version, combined with a particular utility structure, representing the preferences over characteristics of the representative consumer, is shown to generate the classic Singh and Vives (1984) utility function. This simple framework provides us a way to study firms' design choices. We showed that this strategic interaction results in both firms providing the same design. Thus, *horizontal* product differentiation results *minimal.*

To check the robustness of this result we considered two standard costs structures; linear and quadratic. With a linear costs structure, i.e constant marginal costs, our conclusion is that for a sufficiently small marginal cost firms will still provide the same product's design. On the other hand, if marginal cost it too high, no clear-cut conclusion can be derived regarding firms' design choices. Instead, with a quadratic costs structure, i.e variable marginal costs and constant average costs, we discovered that the SPNE of the products' designs subgame remains unchanged: for any possible average cost firms will always select the same product's design. These results confirm that for the current model, the *principle of minimal product differentiation* applies and is sufficiently robust.

The importance of this interesting, yet modest result, hinges on the fact that it arises in a model embodying the main features of a monopolistic market. Interestingly enough, explicit consideration of firms' strategic choices of products' design has not been considered in the literature dealing with monopolistic markets. Thus, the simple setting we proposed may provide an intuition for future study of endogenous product differentiation in monopolistic markets in which firms strategically chose their product's design.

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A SPNE for symmetric "positions"

Symmetric "positions" in the unitary interval $[0,1]$ implies that, in equilibrium, $a_{11}^*+a_{21}^* = 1$. Consider firm $j=1$; then, firm $j=1$'s optimal choice of a_{11}, a_{11}^*, is found by substituting in (10) $a_{21} = 1-a_{11}$ and setting its derivative equal to zero. Firm $j=2$'s optimal choice of a_{21} is found proceeding in a similar way, but substituting $a_{11} = 1-a_{21}$ into (11). The SPNE must then be such that $a_{11}^* + a_{21}^* = 1$. Profit functions, in a symmetric setting, are given by

$$\pi_1(a_{11}) = \left[\frac{2a_{11}+2\sqrt{1-a_{11}^2}}{4-(a_{11}(1-a_{11})+\sqrt{(1-a_{11}^2)(1-(1-a_{11})^2)})^2} - \frac{(a_{11}(1-a_{11})+\sqrt{(1-a_{11}^2)(1-(1-a_{11})^2)})((1-a_{11})+\sqrt{1-(1-a_{11})^2})}{4-(a_{11}(1-a_{11})+\sqrt{(1-a_{11}^2)(1-(1-a_{11})^2)})^2}\right]^2 \quad (25)$$

$$\pi_2(a_{21}) = \left[\frac{2a_{21}+2\sqrt{1-a_{21}^2}}{4-(a_{21}(1-a_{21})+\sqrt{(1-a_{21}^2)(1-(1-a_{21})^2)})^2} - \frac{(a_{21}(1-a_{21})+\sqrt{(1-a_{21}^2)(1-(1-a_{21})^2)})((1-a_{21})+\sqrt{1-(1-a_{21})^2})}{4-(a_{21}(1-a_{21})+\sqrt{(1-a_{21}^2)(1-(1-a_{21})^2)})^2}\right]^2 \quad (26)$$

Maximization of (25) and (26) reveals that, for both firms, the profit-maximizing "position" is the same: $a_{11}^* = a_{21}^* = \bar{a} = \sqrt{77}/10 \approx 0.8775$. Conversely, (25) and (26) reach their minimum at $\underline{a} = 27/125 \approx 0.216008$. Interestingly enough, a firm's optimal strategy corresponds to the rival's positioning at its profit minimizing spot. This is a typical "winner-takes-all" situation. Clearly, this situation cannot constitute a valid *symmetric* SPNE. Thus, we conclude that no pure strategy SPNE exists for the products' selection subgame when *symmetric* "positions" are considered.