ISSN: 2281-1346

UNIVERSITÀ DI PAVIA

Department of Economics and Management

DEM Working Paper Series

Notes and Remarks on Convex and Generalized Convex Functions in Optimization Problems

Giorgio Giorgi
(Università degli Studi di Pavia)

220 (05-24)

Via San Felice, 5
I-27100 Pavia

https://economiaemanagement.dip.unipv.it/it

Notes and Remarks on Convex and Generalized Convex Functions in Optimization Problems

Giorgio Giorgi (*)

(*) Department of Economics and Management - University of Pavia - Via S. Felice, 5 - 27100 PAVIA (Italy). E-mail: giorgio.giorgi@unipv.it

Abstract. In the second section we make some remarks on the use of generalized convexity in Kuhn-Tucker optimality conditions for a mathematical programming problem. In the third section we give a simple proof of a result of O. L. Mangasarian (1988) on the characterization of solution sets of convex programming problems. In the fourth section we make some "critical" comments on invex functions and their use in mathematical programming theory.

Key words: Generalized convexity, optimality conditions, solution sets, invex functions.

AMS subject classifications: 90C30, 90C25, 90C26.

1. Introduction

It is well known that convex functions and generalized convex functions play a key role in mathematical programming. The fact that many practical problems involve functions that are not convex, has produced many generalizations of the notion of convex functions. The class of "generalized convex functions" is by now very wide, perhaps too wide, with reference to the utility in applications of some members of the said functional class.

The aim of the present paper is to point out some non correct statements, previously appeared, on properties of generalized convex functions, and to make some precisions and remarks on properties of convex and generalized convex functions, mainly with reference to optimality conditions for mathematical programming problems. The paper is organized as follows. Section 2 is concerned with some questions on quasiconvex and related functions, in connection with their applications in mathematical programming theory. In Section 3 we give a simple proof of a result of O. L. Mangasarian (1988) on the characterization of solution sets of convex programming. Section 4 is concerned with some "critical" remarks on the class of invex functions and the final Section 5 presents some conclusions.

2. Some questions on quasiconvex and related functions in mathematical programming

S. Mititelu (1974) presents the following result (the original Theorem 2 in Mititelu (1974) is not fully correct).

Theorem 1. Let the functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., m$, and $h_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $j = 1, ..., p$, be differentiable on the open convex set $S \subset \mathbb{R}^n$. Assume that the functions g_i, $i = 1, ..., m$, are quasiconvex on S, and the functions h_j, $j = 1, ..., p$, are quasiconvex and quasiconcave on S. Consider a point $x^0 \in K$, where

$$K = \{x \in S : g(x) \leqq 0,\ h(x) = 0\},$$

and set

$$I(x^0) = \left\{i : g_i(x^0) = 0\right\}.$$

If $u_i \geqq 0$, $i = 1, ..., m$, and $v_j \in \mathbb{R}$, $j = 1, ..., p$, exist such that

$$(x - x^0)^\top \nabla_x \mathcal{L}(x^0, u, v) \geqq 0, \quad \forall x \in K,$$

where

$$\mathcal{L}(x, u, v) = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{j=1}^{p} v_j h_j(x),$$

and:

(a) If for at least one $i \in I(x^0)$, we have $u_i > 0$ and

$$(x - x^0)^\top \nabla g_i(x^0) < 0, \quad \forall x \in K \diagdown \left\{x^0\right\},$$

then x^0 is a strict local minimum point of f over K.

(b) If it holds

$$K \cap \left\{x \in \mathbb{R}^n : (x - x^0)^\top \nabla f(x^0) = 0\right\} = \left\{x^0\right\},$$

then x^0 is a strict local minimum point of f over K.

(c) If the function f is quasiconvex on S and $\nabla f(x^0) \neq 0$, then x^0 is a global minimum point of f over K.

For what concerns the proof of point (a) of the previous theorem, the author arrives to the conclusion by proving that the following inequality holds true:

$$(x - x^0)^\top \nabla f(x^0) > 0, \quad \forall x \in K \diagdown \left\{x^0\right\}. \tag{1}$$

The same relation is used to prove point (b). The result sub (a) is correct, but it needs a specific justification, as the author is not clear on the conclusion of the related proof. If we have an optimization problem of the type

$$(P_0): \qquad \min f(x), \quad x \in S \subset \mathbb{R}^n,$$

we can state the following result (see, e. g., Giorgi, Jiménez and Novo (2023), Hestenes (1975)).

Theorem 2. Let $f : S \longrightarrow \mathbb{R}$ be differentiable on $S \subset \mathbb{R}^n$. A sufficient condition for $x^0 \in S$ to be a strict local minimum of $f(x)$ on S (i. e. a strict local solution of (P_0)) is

$$y^\top \nabla f(x^0) > 0, \quad \forall y \in T(S, x^0) \diagdown \{0\},$$

where $T(S, x^0)$ is the *Bouligand tangent cone* (or *contingent cone*) to S at x^0, i. e.

$$T(S, x^0) = \left\{ \begin{array}{c} y \in \mathbb{R}^n : \exists \{x^n\} \subset S, \ \exists \{\lambda_n\} \subset \mathbb{R}_+, \text{ such that} \\ \lim_{n \longrightarrow +\infty} x^n = x^0, \ \lim_{n \longrightarrow +\infty} \lambda_n (x^n - x^0) = y \end{array} \right\}.$$

When $S \subset \mathbb{R}^n$ is a *convex set,* it holds

$$T(S, x^0) = cl \left\{ cone(S - x^0) \right\},$$

and the thesis of Theorem 2 becomes

$$(x - x^0)^\top \nabla f(x^0) > 0, \quad \forall x \in S, \ x \neq x^0.$$

Under the assumptions of Theorem 1, the feasible set K is convex, therefore the thesis sub (a) follows. The same remarks apply to Mititelu (1987) and to Giorgi and Mititelu (1983). Obviously, if $f(x)$ is *quasiconvex* on S, from (1) we obtain

$$(x - x^0)^\top \nabla f(x^0) > 0 \implies f(x) > f(x^0), \quad \forall x \in K \diagdown \left\{x^0\right\},$$

i. e. x^0 is a strict global minimizer of f over K. Furthermore, we note that the condition sub (a) of Theorem 1 is verified if that constraint $g_i(x)$, $i \in I(x^0)$, for which it holds $u_i > 0$, is *strictly pseudoconvex.* We recall that $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is strictly pseudoconvex on the open convex set S, if for every $x, \bar{x} \in S$, $x \neq \bar{x}$,

$$(x - \bar{x})^\top \nabla f(\bar{x}) \geqq 0 \implies f(x) > f(\bar{x}),$$

or equivalently,

$$f(x) \leqq f(\bar{x}) \implies (x - \bar{x})^\top \nabla f(\bar{x}) < 0.$$

We take the opportunity to recall that, contrary to a famous claim of J. L. Lagrange, a *local ray minimum point* x^0 is *not* necessarily a local minimum point, as the following well known counterexample, due to the Italian mathematician G. Peano, shows. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = (y - x^2)(y - 2x^2).$$

The point $x^0 = (0, 0)$ is a local ray minimum point over the plane, but not a local minimum point for f. See, e. g., Apostol (1974), Giannessi (2005), Qi (2001), Thompson and Parke (1973). Always with reference to this subject, Thompson and Parke (1973) prove the following two results. We recall the next definition.

Definition 1. The function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is *semistrictly quasiconvex* on the convex set $S \subset \mathbb{R}^n$, if

$$f(\lambda x^1 + (1-\lambda)x^2) < \max\left\{f(x^1),\ f(x^2)\right\},$$

for every $x^1, x^2 \in S$, with $f(x^1) \neq f(x^2)$, and for every $\lambda \in (0,1)$,
or equivalently, if

$$f(x^1) > f(x^2) \implies f(x^1) > f(x^1 + \lambda(x^2 - x^1)),\quad \forall x^1, x^2 \in S,\ \forall \lambda \in (0,1).$$

Theorem 3. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be quasiconvex on the convex set $S \subset \mathbb{R}^n$; then any local ray minimum point of f is a local minimum point of f over S.

Theorem 4. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is semistrictly quasiconvex on the convex set $S \subset \mathbb{R}^n$, then $x^0 \in S$ is a global minimum point of S over S if and only if x^0 is a local ray minimum point of f over S.

Remark 1. On the grounds of Theorem 4, we can assert that if $K = \{x \in S \subset \mathbb{R}^n : g_1(x) \leqq 0, ..., g_m(x) \leqq 0\}$, and every $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, is semistrictly quasiconvex on the convex set $S \subset \mathbb{R}^n$, if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is semistrictly quasiconvex on S, then $x^0 \in K$ is the global minimum point of f over K if and only if x^0 is a local ray minimum point of f over K.

Result (c) of Theorem 1 is proved by S. Mititelu by making reference to the paper of Kortanek and Evans (1967), where indeed a similar result is proved. In a nonlinear programming problem of the type

$$(P): \qquad \min f(x), \quad x \in K,$$

where $K = \{x \in S \subset \mathbb{R}^n : g_i(x) \leqq 0,\ i = 1, ..., m\}$, with f and every g_i, $i = 1, ..., m$, real-valued functions differentiable on the open convex set $S \subset \mathbb{R}^n$, the fact that the quasiconvexity of f and g_i, $i = 1, ..., m$, does not imply that the related Karush-Kuhn-Tucker conditions are sufficient conditions for global minimum of a feasible point x^0 of (P), had already been shown by Arrow and Enthoven (1961). In order to obtain sufficient conditions, these authors assume, besides the requirement $\nabla f(x^0) \neq 0$ ($f(x)$ quasiconvex), that f is twice-continuously differentiable in a neighbourhood of x^0. Arrow and Entoven (1961) in their pioneering paper, give a quite intricate proof of their proposition. Also Ferland (1972) assumes, besides $\nabla f(x^0) \neq 0$, that the objective function of (P) is twice differentiable on S. In Mukherji (1989) the twice differentiability of $f(x)$ is not assumed, but it is imposed on the constraints a Slater-type condition, which in fact plays no role. Note that twice differentiability is not required by Kortanek and Evans (1967), Giorgi (1984), Guignard (1969). We present the following simple proof.

Theorem 5. Consider the above problem (P) and let $x^0 \in K$, let $f : S \longrightarrow \mathbb{R}$ be differentiable and quasiconvex on the open convex set $S \subset \mathbb{R}^n$, let every $g_i : S \longrightarrow \mathbb{R}$ be differentiable and quasiconvex on S; let be $\nabla f(x^0) \neq 0$ and let the Karush-Kuhn-Tucker conditions for (P) be satisfied at x^0. Then x^0 solves (P).

Proof. We first prove that, under the above assumptions, we have, for all x, x^0 :

$$f(x) - f(x^0) < 0 \implies (x - x^0)^\top \nabla f(x^0) < 0. \tag{2}$$

Suppose $f(x) < f(x^0)$ and choose $\alpha > 0$ so small that $f(x + \alpha \nabla f(x^0)) \leqq f(x^0)$. Then we have, thanks to the quasiconvexity of $f(x)$,

$$(x + \alpha \nabla f(x^0) - x^0)^\top \nabla f(x^0) \leqq 0$$

or

$$(x - x^0)^\top \nabla f(x^0) \leqq -\alpha \nabla f(x^0)^\top \nabla f(x^0) < 0,$$

being $\nabla f(x^0) \neq 0$ and $\alpha > 0$. So, we have proved (2).

As every constraint $g_i(x)$, $i \in I(x^0)$, is quasiconvex and differentiable, we have, for every $i \in I(x^0)$ and every $x \in K$:

$$g_i(x) - g_i(x^0) \leqq 0 \implies (x - x^0)^\top \nabla g_i(x^0) \leqq 0.$$

Therefore, with $\lambda_i \geqq 0$, $\forall i \in I(x^0)$, we have, for every $x \in K$:

$$\sum_{i \in I(x^0)} \lambda_i (x - x^0)^\top \nabla g_i(x^0) \leqq 0.$$

With $\lambda_i = 0$, $\forall i \notin I(x^0)$, we can write

$$\sum_{i=1}^{m} \lambda_i (x - x^0)^\top \nabla g_i(x^0) \leqq 0, \quad \forall x \in K. \tag{3}$$

By the Karush-Kuhn-Tucker conditions, we have

$$(x - x^0)^\top \left[\nabla f(x^0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^0) \right] = 0,$$

that is

$$(x - x^0)^\top \nabla f(x^0) + \sum_{i=1}^{m} \lambda_i (x - x^0)^\top \nabla g_i(x^0) = 0.$$

But, for every $x \in K$, relation (3) holds, so, for every $x \in K$:

$$(x - x^0)^\top \nabla f(x^0) \geqq 0.$$

By taking the contrapositive law of (2) we get, $\forall x \in K$,

$$(x - x^0)^\top \nabla f(x^0) \geqq 0 \implies f(x) \geqq f(x^0). \quad \Box$$

Nonlinear programming problems with a quasiconvex objective functions have been considered also by Bector, Chandra and Bector (1988). However, their Theorem 3.1 is not correct, as it implies that in the related Fritz John conditions, the multiplier $\bar{y}_0$, associated with the gradient of the objective function (evaluated at the feasible point x^0), is *positive* (i. e. the said theorem works if the Karush-Kuhn-Tucker conditions are assumed to hold). The same remark is true for Theorems 4.1, 4.2 and 4.3 of the same paper. See Giorgi (1994).

Bector and Singh (1991) introduce the notion of *B-vex functions.*

Definition 2. A function $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, with X a convex set, is called *B-vex,* with respect to (b_1, b_2), if for every $x, u \in X$, and for every $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)u) \leqq b_1(x, u, \lambda)f(x) + b_2(x, u, \lambda)f(u),$$

where $b_1(x, u, \lambda) + b_2(x, u, \lambda) = 1$, $b_1(x, u, \lambda) \geqq 0$, $b_2(x, u, \lambda) \geqq 0$.

We can see that a function is B-vex if and only if it is quasiconvex. See, e. g., Li, Dong and Liu(1997), Giorgi and Molho (1992). By this specification the properties of quasiconvex functions can be investigated further by means of B-vex functions; without the said specification, Definition 2 may increase misunderstanding and confusion. Other questionable papers on this subject are Bector, Suneja and Lalitha (1993), and Rueda, Singh and Bector (1992).

Let us consider again problem (P), where $f(x)$ and every $g_i(x)$, $i = 1, ..., m$, are *convex* on the convex set $S \subset \mathbb{R}^n$. Under these assumptions, Künzi, Krelle and Oettli (1966) prove the following result.

- If $x^0 \in K$ is an optimal solution of problem (P), then x^0 is also an optimal solution of the "reduced problem"

$$\begin{array}{lcl} (P_1) & : & \min f(x) \\ \text{subject to:} & & g_i(x) \leqq 0,\ i \in I(x^0),\ x \in S. \end{array}$$

The above result holds under weaker assumptions.

Theorem 6. Consider problem (P), where $S \subset \mathbb{R}^n$ is a convex set, $x^0 \in K$, $f(x)$ is semistrictly quasiconvex on S and the functions $g_i(x)$, $i \in I(x^0)$, are quasiconvex on S, the functions $g_i(x)$, $i \notin I(x^0)$, are continuous at x^0. Then, if x^0 is a solution of (P), it is also a solution of the "reduced problem" (P_1).

Proof. Absurdly suppose that there exists $\bar{x} \in S$ such that $g_i(\bar{x}) \leqq 0$, $\forall i \in I(x^0)$ and $f(\bar{x}) < f(x^0)$. Then, for every $\lambda \in (0, 1)$ we have $x_\lambda \equiv \lambda\bar{x} + (1 - \lambda)x^0 \in S$

$$f(x_\lambda) < f(x^0), \quad g_i(x_\lambda) \leqq \max\left\{g_i(\bar{x}), g_i(x^0) = 0\right\} \leqq 0,\ \forall i \in I(x^0). \qquad (4)$$

For every $i \notin I(x^0)$ we have $g_i(x^0) < 0$, and as $g_i(x)$, $i \notin I(x^0)$, is continuous at x^0, we have $g_i(x_\lambda) < 0$, $i \notin I(x^0)$, for λ sufficiently small. Then, x_λ satisfies the initial restrictions and (4) contradicts the optimality of x^0. $\square$

Remark 2. The converse of the above result is not true, as the "reduced problem" can violate some of the dropped constraints of the original problem. The version of Künzi, Krelle and Oettli (1966) has been rediscovered by Moeseke (1974) and used by this author in proving a saddle point result for a convex programming problem, with homogeneous functions, but without assuming the *Slater constraint qualification.*

Obviously, generalized convex functions have been widely used also in obtaining optimality and duality results in *vector optimization problems.* We point out the following remark concerning the paper of Cambini and Martein (2001). In this paper Theorem 2 and Theorem 12 should not be correct, as it appears that $H_f(x^0)$ is the usual Hessian matrix of a scalar function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, evaluated at x^0. But then, the quantity $y^\top H_f(x^0)y$, with $y \in \mathbb{R}^n$, is a scalar quantity, not a vector. If $f''(x^0)$ denotes the second-order Fréchet derivative of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, then $f''(x^0)(y, y)$ obviously must be a vector of $\mathbb{R}^m$, not a scalar. A solution could be to define $y^\top H_f(x^0)y$ as the vector of $\mathbb{R}^m$ whose i-th component is $y^\top H_{f_i}(x^0)y$, $i = 1, ..., m$, i. e.

$$y^\top H_f(x^0)y = \begin{bmatrix} y^\top H_{f_1}(x^0)y \\ y^\top H_{f_2}(x^0)y \\ \vdots \\ y^\top H_{f_m}(x^0)y \end{bmatrix}.$$

3. A simple proof of a result of O. L. Mangasarian

O. L. Mangasarian (1988) has given a nice characterization of the solution set of the mathematical programming problem

$$\min f(x), \quad x \in S,$$

where S is a convex set of $\mathbb{R}^n$ and f is a twice continuously differentiable convex function on some open convex set of $\mathbb{R}^n$ containing S. Let S^* be the (nonempty) solution set of the above problem, i. e.

$$S^* = \arg\min_{x \in S} f(x),$$

and let x^* be any point of S^*. Then it holds

$$S^* = \bar{S} \equiv \left\{ x \in S : (x - x^*)^\top \nabla f(x^*) = 0; \quad \nabla f(x) = \nabla f(x^*) \right\}. \tag{5}$$

For other proofs of the above result, see Burke and Ferris (1991), Ketabchi and Ansari-Piri (2007). The result of Mangasarian can be proved in a simple way, under the assumption of once differentiability of $f(x)$.

Theorem 7. Consider the problem

$$\min f(x), \quad x \in S,$$

where S is a convex set of $\mathbb{R}^n$ and $f(x)$ is a differentiable convex function on an open convex set of $\mathbb{R}^n$ containing S. Let be $x^* \in \arg\min_{x \in S} f(x) \equiv S^*$. Then relations (5) hold.

Proof. Let x^* be an optimal solution of the above problem, that is $x^* \in S^*$, and $f(x) = f(x^*)$. Since S^* is convex, it follows that

$$f(\lambda x + (1-\lambda)x^*) = f(x) = f(x^*) \tag{6}$$

for all $\lambda \in (0,1)$. From the property

$$(x^1 - x^2)^\top \nabla f(x^2) \leqq f(x^1) - f(x^2), \quad \forall x^1, x^2 \in S,$$

it follows that

$$\lambda(x - x^*)^\top \nabla f(x^*) \leqq f(x_\lambda) - f(x^*) \leqq \lambda(x - x^*)^\top \nabla f(x_\lambda), \tag{7}$$

where $x_\lambda = \lambda x + (1-\lambda)x^*$. From (6) and (7) it follows that

$$(x - x^*)^\top \nabla f(x^*) \leqq 0. \tag{8}$$

If we assume that

$$(x - x^*)^\top \nabla f(x^*) < 0, \tag{9}$$

then we would have

$$(x - x^*)^\top \nabla f(x_\lambda) < 0 \tag{10}$$

for a small λ. The relation (10) contradicts the second inequality in (7) and hence (9) does not hold. Thus, from (8) the first equality of (5) follows. In order to prove the second equality of (5), let us consider the convex function

$$\varphi(y) = f(y) - (y - x^*)^\top \nabla f(x^*),$$

where $y \in \mathbb{R}^n$. From the first equality of (5) it follows that $\varphi(x) = f(x)$. Since $\varphi(x^*) = f(x^*)$, from (6) it follows that $\varphi(x) = \varphi(x^*)$. Since $\nabla\varphi(x^*) = 0$, it follows that x^* is an unconstrained minimizer of the function φ. From the fact that $\varphi(x) = \varphi(x^*)$, it follows that x is also an unconstrained minimizer and hence $\nabla\varphi(x) = 0$. Since

$$\nabla\varphi(x) = \nabla f(x) - \nabla f(x^*),$$

the second equality of (5) follows easily. The proof is completed by noting that since $f(x)$ is a convex function, we have

$$(x - x^*)^\top \nabla f(x^*) \leqq f(x) - f(x^*) \leqq (x - x^*)^\top \nabla f(x). \tag{11}$$

From (5) and (11) it follows that $f(x) = f(x^*)$ and hence $x \in S^*$. $\square$

The previous result of Mangasarian has been generalized towards various directions. For instance, Jeyakumar and Yang (1995) assume that $f(x)$ is *pseudolinear* (i. e. both pseudoconvex and pseudoconcave) on the convex set $S \subset \mathbb{R}^n$. Under this assumption, these authors prove that $S^* = \arg\min_{x \in S} f(x)$ is given by the first relation of (5), i. e. we have

$$S^* = \bar{S}_1 \equiv \left\{ x \in S : (x - x^*)^\top \nabla f(x^*) = 0 \right\}.$$

The same authors, always under the assumption of pseudolinearity of $f(x)$, prove also the following characterizations:

$$S^* = \bar{S}_2 = \left\{ x \in S : (x - x^*)^\top \nabla f(x) = 0 \right\};$$

$$S^* = \bar{S}_3 = \left\{ x \in S : (x - x^*)^\top \nabla f((1 - \alpha)x + \alpha x^*) = 0, \ \forall \alpha \in [0, 1] \right\}.$$

The results of Jeyakumar and Yang (1995) have been generalized by Ansari, Schaible and Yao (1999), who assume that $f(x)$ is *η-pseudolinear on S*, i. e. both $f(x)$ and $-f(x)$ are *η-pseudoinvex,* with respect to the same vector-valued function $\eta(x, y)$. See the next section. Under this assumption and other additional assumptions, the said authors prove the following results:

$$S^* = \arg\min_{x \in S} f(x) = \bar{S}_4 \equiv \left\{ x \in S : \eta(x^*, x)^\top \nabla f(x) = 0 \right\};$$

$$S^* = \bar{S}_5 \equiv \left\{ x \in S : \eta(x^*, x)^\top \nabla f(x^*) = 0 \right\}.$$

The characterization of S^* under the assumption that $f(x)$ is *pseudoconvex* on S is not a trivial problem and has been treated by Castellani and Giuli (2012) and by Ivanov (2013a,b;2019). The last author gives, under the assumption that $f(x)$ is pseudoconvex on S, the following results:

$$S^* = \arg\min_{x \in S} f(x) = \hat{S} \equiv \left\{ \begin{array}{c} x \in S : (x - x^*)^\top \nabla f(x^*) = 0; \\ \exists p(x) > 0 : \nabla f(x) = p(x) \nabla f(x^*) \end{array} \right\};$$

$$S^* = \hat{S}_1 = \left\{ x \in S : (x - x^*)^\top \nabla f(x^*) \leqq 0; \ \exists p(x) > 0 : \nabla f(x) = p(x) \nabla f(x^*) \right\}.$$

For other related results, see Wu and Wu (2006), Wu (2008), Lalitha and Mehta (2009), Son and Kim (2014). The results of Yang (2009), always with reference to the assumption of pseudoconvexity of $f(x)$, characterize $S^* =$

$\arg\min_{x \in S} f(x)$ in terms of the gradient of $f(x)$, evaluated at $x \in S$. This author gives the following characterizations of S^* :

$$A) \quad S^* = \left\{x \in S : (x - x^*)^\top \nabla f(x) = 0\right\};$$

$$B) \quad S^* = \left\{x \in S : (x - x^*)^\top \nabla f(x) \leqq 0\right\}.$$

The above results sub A) and B) are a corollary of a more general theorem of Yang (2009) involving *invex functions* (see the next section). We give a direct and "autonomous" proof of the result sub A).

Proof of A).

Let us denote by S_1 the set $\left\{x \in S : (x - x^*)^\top \nabla f(x) = 0\right\}$. The inclusion $S^* \subset S_1$ follows from the fact that $f(x)$ is pseudoconvex and hence quasiconvex. Therefore the set S^* is a convex set (see, e. g., Avriel, Diewert, Schaible and Zang (1988), Martos (1975); this result follows easily from the convexity of the lower level sets of $f(x)$). But if S^* is a convex set, and if $x, x^* \in S^*$, then we have, for every $\lambda \in [0,1]$, $\lambda x + (1-\lambda)x^* \in S^*$, that is, the function $\varphi(\lambda) \equiv f(\lambda x + (1-\lambda)x^*)$ is constant on $[0,1]$ and hence, for every $\lambda \in (0,1)$, we have $0 = \varphi'(\lambda) = (x - x^*)^\top \nabla f(\lambda x + (1-\lambda)x^*)$. Letting $\lambda \longrightarrow 0$ and $\lambda \longrightarrow 1$, respectively, we obtain

$$S^* \subset \left\{x \in S : (x - x^*)^\top \nabla f(x^*) = 0\right\}$$

and

$$S^* \subset S_1.$$

For the opposite inclusion, let $x \in S_1$, relation we rewrite in the form

$$(x^* - x)^\top \nabla f(x) = 0, \quad x \in S. \tag{12}$$

If $f(x^*) < f(x)$, being $f(x)$ pseudoconvex, we have, by the definition of pseudoconvex functions, $(x^* - x)^\top \nabla f(x) < 0$, in contradiction with (12). Hence it must hold $f(x^*) = f(x)$, i. e. $x \in S^*$. $\square$

Always under the assumption that $f(x)$ is pseudoconvex on the convex set $S \subset \mathbb{R}^n$, we have also the following strict inclusion

$$\bar{S} = \left\{x \in S : (x - x^*)^\top \nabla f(x^*) = 0;\ \nabla f(x) = \nabla f(x^*)\right\} \subset S^* \equiv \arg\min_{x \in S} f(x).$$

Indeed, consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(x,y) = y/x$, which is pseudoconvex on $S = \{(x,y) : x > 0, y \geqq 0\}$. We have $S^* = \{(x,0) : x > 0\}$, however $\nabla f(x,0) = \begin{bmatrix} 0 \\ 1/x \end{bmatrix}$, which is not constant over $\bar{S}$.

An interesting result on the subject of the present section is given by Horst (1971, 1972), who takes into consideration the so-called *convexifiable functions* or *convex range transformable functions* (in Horst (1971), (1972) these functions are called “Mittelbar konvexe Funktionen”). See also Avriel and Zang (1974), Avriel, Diewert, Schaible and Zang (1988), Horst (1984), Mond (1983), Zang (1981).

Definition 3. Let $f : C \longrightarrow \mathbb{R}$ be defined on the convex set $C \subset \mathbb{R}^n$, and denote by $I = I_f(C)$ the range of $f(x)$, i. e. the image of C under f. Then $f(x)$ is said to be *convexifiable* or *convex range transformable* or briefly *h-convex,* if there exists a continuous strictly monotone increasing function $h : I \longrightarrow \mathbb{R}$, such taht $h\,[f(x)]$ is convex over C, that is

$$h\left[f(\lambda x^1 + (1-\lambda)x^2)\right] \leqq \lambda h\left[f(x^1)\right] + (1-\lambda) h\left[f(x^2)\right], \quad \forall x^1, x^2 \in C,\ \forall \lambda \in [0,1]\,,$$

that is

$$f(\lambda x^1 + (1-\lambda)x^2) \leqq h^{-1}\left[\lambda h(f(x^1)) + (1-\lambda) h f(x^2))\right], \ \forall x^1, x^2 \in C,\ \forall \lambda \in [0,1]\,.$$

Examples of convexifiable functions are the *r-convex functions,* introduced by Avriel (1972) and Martos (1975), obtained by taking $h(x) = e^{rx}$, $x \in \mathbb{R}$, $r \neq 0$. Then $h^{-1}(x) = \log x^{1/r}$, $x > 0$. Therefore we have

$$f(\lambda x^1 + (1-\lambda)x^2) \leqq \begin{cases} \log\left[\lambda e^{rf(x^1)} + (1-\lambda) e^{rf(x^2)}\right]^{\frac{1}{r}}, & r \neq 0 \\ \lambda f(x^1) + (1-\lambda) f(x^2), & r = 0, \end{cases}$$

for every $x^1, x^2 \in C$, $\forall \lambda \in [0,1]$.

Taking $h(x) = x^p$, $x > 0$, $p \neq 0$, we have the *power convex functions,* or *p-convex functions,* studied by Avriel (1972) and Lindberg (1981). We have $h^{-1}(x) = x^{1/p}$, $x > 0$, and hence

$$f(\lambda x^1 + (1-\lambda)x^2) \leqq \begin{cases} \left[\lambda f^p(x^1) + (1-\lambda) f^p(x^2)\right]^{\frac{1}{p}}, & p \neq 0 \\ f^{\lambda}(x^1) \cdot f^{(1-\lambda)}(x^2), & p = 0. \end{cases}$$

It can be proved that every h-convex function is *semistrictly quasiconvex,* but in general the converse does not hold. Moreover, if f and h are differentiable, h-convex functions are *pseudoconvex,* but in general the converse does not hold. Some basic properties typical of convex functions continue to hold for the class of convexifiable functions; for example, every local minimum of h-convex functions is also a global one, and if f and h are differentiable, then every stationary point of f is a global minimum point over C (on the other hand, in this case f is pseudoconvex). In particular, the set of global minimizers of h-convex functions is a convex set and every h-convex function is continuous on the interior of its domain C. Horst (1971, 1972) proves also that the *saddle point conditions* continue to hold for the class of h-convex functions and, for what concerns the problem of the present section, this author proves the following result.

Theorem 8. Let $f : C \longrightarrow \mathbb{R}$ be convexifiable and differentiable on the open convex set $C \subset \mathbb{R}^n$, containing the convex set S; let be $h'\,[f(x)] \neq 0$ for all $x \in C$. Then, denoting by $S^* \equiv \arg\min_{x \in S} f(x)$, the set of minimizers of f over S, it holds

$$S^* = \left\{ x \in S : (x - x^*)^\top \nabla f(x^*) = 0; \quad \nabla f(x) = \nabla f(x^*) \right\}.$$

Hence, under the said assumptions, we have the same characterization of S^*, proved by Mangasarian (1988) for the class of convex functions (Theorem 7), i. e. $S^* = \bar{S}$.

4. The "vexata quaestio" of invex functions

Invex functions are a type of generalized convex functions which received a lot of attentions in the last 40 years. The interest in invex and related functions is essentially due to the fact that this class of functions can be used to extend the sufficiency of the Karush-Kuhn-Tucker conditions and the duality theory of the class of convex programs to a more general class of optimization problems. See Hanson (1981). For some references on this subject (not up to date), see, e. g., Mishra and Giorgi (2008). These functions have also produced some "critical" opinions and papers. The quite recent paper of Zalinescu (2014) is an important example. Also the author of the present paper has been concerned with the class of invex functions, both in the differentiable case and in the nonsmooth case, both for scalar optimization problems and for vector optimization problems. See, for example, Giorgi (2008a, b; 1990), Mishra and Giorgi (2008), Giorgi and Guerraggio (1996, 1998a, b; 2000) Giorgi, Jiménez and Novo (2009), Giorgi and Molho (1992), Giorgi and Rueda (2009).

At present, my opinion is that invex functions are an interesting class of generalized convex functions, with reference to theoretical-analytical topics; less interesting, with reference to their applications.

Definition 4. Let be given the open convex set $C \subset \mathbb{R}^n$ and let $f : C \longrightarrow \mathbb{R}$ be differentiable on C. Then $f(x)$ is *invex* on C if there exists a vector function (called also "kernel") $\eta(x, u) : C \times C \longrightarrow \mathbb{R}^n$, such that, for every x, u :

$$f(x) - f(u) \geqq \eta(x, u)^\top \nabla f(u). \tag{13}$$

See Hanson (1981). Subsequently, other related classes have been introduced. See, for example, Kaul and Kaur (1985), Pini (1991).

Definition 5. The differentiable function $f : C \longrightarrow \mathbb{R}$ is *pseudoinvex* on C if there exists $\eta(x, u) : C \times C \longrightarrow \mathbb{R}^n$, such that for every $x, u \in C$:

$$\eta(x, u)^\top \nabla f(u) \geqq 0 \implies f(x) \geqq f(u).$$

Definition 6. The differentiable function $f : C \longrightarrow \mathbb{R}$ is *quasiinvex* on C if there exists $\eta(x,u) : C \times C \longrightarrow \mathbb{R}^n$, such that for every $x, u \in C$:

$$f(x) \leqq f(u) \implies \eta(x,u)^\top \nabla f(u) \leqq 0.$$

Remark 3. We remark that, by Definition 6, every differentiable function is quasiinvex: it is sufficient to take $\eta(x,u)$ identically equal to the zero vector of $\mathbb{R}^n$. However, we should exclude this "extreme" case, as the kernel function $\eta(x,u)$ was introduced as a generalization of the linear function $(x-u)$, which appears in the classical definitions of (differentiable) convex, pseudoconvex and quasiconvex functions.

Consider the following mathematical programming problem:

$$(P): \qquad \min f(x), \quad x \in K = \{x \in \mathbb{R}^n : g_i(x) \leqq 0, \ i = 1, ..., m\}$$

and let $x^0 \in K$. Suppose that $f(x)$ and every $g_i(x)$, $i \in I(x^0)$, are invex functions with respect to the *same* kernel $\eta(x,u)$. If there exist λ_i, $i = 1, ..., m$, such that

$$\nabla f(x^0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^0) = 0;$$

$$\lambda_i \geqq 0, \ i = 1, ..., m;$$

$$\lambda_i g_i(x^0) = 0, \ i = 1, ..., m,$$

then x^0 is a solution of (P).

The proof is easy: for every $x \in K$ we have

$$f(x) - f(x^0) \geqq \eta(x.x^0)^\top \nabla f(x^0) = -\sum_{i \in I(x^0)} \lambda_i \eta(x.x^0)^\top \nabla g_i(x^0) \geqq$$

$$\geqq -\sum_{i \in I(x^0)} \lambda_i (g_i(x) - g_i(x^0)) \geqq 0.$$

Consequently, the thesis follows.

A simple but interesting characterization of invex functions is given in the following basic result, due to Ben-Israel and Mond (1986) and to Craven and Glover (1985).

Theorem 9. A differentiable function $f : C \longrightarrow \mathbb{R}$ on the open convex set $C \subset \mathbb{R}^n$ is invex on C if and only if every stationary point is a global minimum point.

It turns out that the class of invex functions coincides with the class of pseudoinvex functions (not necessarily by choosing the same kernel function

η). However, some authors present the class of pseudoinvex functions as a true generalization of the class of invex functions.

We remark also that Example 3 of Giorgi and Molho (1992) is not correct, even if the related assertion is true, that is the set of minimum points of an invex function in general is *not* convex. A correct example is the following one. Consider

$$f(x,y) = y(x^2-1)^2$$

defined on $X = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. The set of all minimum points is given by

$$\arg\min f(x,y) = \{(1,y) : y > 0\} \cup \{(-1,y) : y > 0\}$$

which is *not* a convex set.

Note that the above points are stationary points of f, so f is invex; however, by setting $a = (-1,1)$ and $b = (\frac{1}{2},1)$, we have $f(a) = 0 < f(b) = \frac{9}{16}$ and $(a-b)^\top \nabla f(b) = (-\frac{3}{2},0)^\top(-\frac{3}{2},\frac{9}{16}) = \frac{9}{4} > 0$, so that f is *not* quasiconvex and hence also *not* pseudoconvex.

Moreover, it appears that, contrary to what holds for convex functions, if the set $C_0 \subset C \subset \mathbb{R}^n$ is *not* open, it is not longer true for an invex function on C_0, that a local minimum point is also a global minimum point. Consider again the function

$$f(x,y) = y(x^2-1)^2$$

and the set $C_0 = \{(x,y) \in \mathbb{R}^2 : x \geqq -\frac{1}{2},\ y \geqq 1\}$. Every stationary point of f on C_0 is a global minimum point on C_0, and therefore f is invex on C_0. The point $(-\frac{1}{2},1)$ is a local mimimum point of f on C_0, with $f(-\frac{1}{2},1) = \frac{9}{16}$, but the global minimum on C_0 is $f(1,y) = 0$.

We have previously remarked that in sufficient Kuhn-Tucker optimality conditions for a nonlinear programming problem (P), but also for a multiobjective programming problem, all functions involved in the said problems must be invex *with respect to the same kernel function* η. This fact may be viewed as a serious restriction to the application of invex functions to optimality conditions; an interesting result of Martinez-Legaz (2009) clarifies the situation.

Theorem 10 (Martinez-Legaz). Let $f_1, ..., f_p$ be differentiable functions defined on an open (convex) subset C of $\mathbb{R}^n$. The following statements are equivalent.

(i) The functions $f_1, ..., f_p$ are invex with respect to the same kernel η.

(ii) The functions $\sum_{i=1}^p \lambda_i f_i$, $\lambda_1 \geqq 0, ..., \lambda_p \geqq 0$, are invex with respect to the same kernel η.

(iii) The functions $\sum_{i=1}^p \lambda_i f_i$, $\lambda_1 \geqq 0, ..., \lambda_p \geqq 0$, are invex.

(iv) For every $\lambda_1 \geqq 0, ..., \lambda_p \geqq 0$, every stationary point of $\sum_{i=1}^p \lambda_i f_i$, is a global minimum point.

The requirement that in sufficient Kuhn-Tucker optimality conditions, the kernel function η must be the same for all functions involved in the problem, remains a rather stringent condition and this fact rises the question whether invexity is indeed a "true" generalization of convexity. Take into consideration the following examples of Phu (2004a). See also for other critical considerations Phu (2004b, c). Let $\varphi(x) = x_1 - (x_2)^2$, $\psi(x) = -x_1$, with $(x_1, x_2) \in \mathbb{R}^2$.

These functions are continuously differentiable on $\mathbb{R}^2$. The functions φ and $-\varphi$ are invex on $\mathbb{R}^2$ with respect to the same kernel

$$\eta(x, y) = x - y + r(x, y),$$

with $r(x, y) = (-(x_2 - y_2)^2, 0)$, for all $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$. Let us consider the problems

$$(P_1) \quad : \qquad \min \varphi(x), \text{ subject to: } \psi(x) \leqq 0.$$
$$(P_2) \quad : \qquad \min \psi(x), \text{ subject to: } \varphi(x) \leqq 0.$$

For $x^* = (0, 0) \in \mathbb{R}^2$ the Kuhn-Tucker conditions are satisfied for both problems. However, x^* is not a local minimizer (and obviously not a global minimizer) of problem (P_1) and the same point is not a local minimizer of problem (P_2). Therefore, the invexity of φ and the convexity of ψ are not sufficient for a Kuhn-Tucker point of (P_1) and (P_2) to be a minimizer. The situation may occur even if ψ is a strictly convex function. Take, for instance, the function

$$\psi(x) = -x_1 + \frac{1}{2}(x_1^2 + x_2^2).$$

Note that in fact $\varphi(x)$ is a concave function and that there exists no function η such that both functions φ and ψ are invex with respect to that η, otherwise $x^* = (0, 0)$ would be a minimizer of (P_1) and of (P_2), as it results from Hanson (1981).

We remark that the functions introduced by Heal (1984) and the ones considered by Rueda (1989) are strictly included in the class of invex functions. See, e. g., Giorgi and Molho (1992) and Hanson and Mond (1987a). Indeed, the functions introduced by Heal (1984) and Rueda (1989) are special classes of (differentiable) *convexifiable functions* (see the previous section). It is therefore immediate their inclusion in the class of invex functions. The reverse implication does not hold, as differentiable convexifiable functions are pseudoconvex, but the converse does not hold. We remark also that the class of functions introduced, as a generalization of invex functions, by Hanson and Mond (1982), coincides in fact with the class of invex functions. See Craven and Glover (1985), Caprari (2003). The above class of functions, subsequently called also *F-functions,* have attracted several authors, all of them with the convinction that this class is wider than the class of invex functions. The list is too long to be reported here; a partial list is given by Caprari (2003). Obviously, all these generalizations which in fact are not generalizations, produce only confusion.

For what regards the relationships of invex functions with other generalized convex functions, see, e. g., Ben-Israel and Mond (1986), Giorgi (1990), Giorgi and Molho (1992), Pini (1991). Here we prove the following result.

Theorem 11. Let $f : C \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable and quasiconvex on the open convex set C. Then f is invex on C if and only if f is pseudoconvex on C.

Proof.
(a) Let us assume that for a point $x^0 \in C$ we have $\nabla f(x^0) = 0$. Then, if f is invex, x^0 is a global minimizer and, being also quasiconvex, then f is also pseudoconvex. If f is pseudoconvex, then $\nabla f(x^0) = 0$ implies that x^0 is a global minimizer and hence, f is invex.

(b) Let us assume that $\nabla f(x^0) \neq 0$ for all $x \in C$. Then, if f is quasiconvex, it is also pseudoconvex (see, e. g., Cambini and Martein (2009)) and hence f is also invex. Conversely, an invex function without stationary points, being quasiconvex, has to be also pseudoconvex. □

On the grounds of the above result, in other words, the intersection between the class of invex functions and the class of (differentiable) quasiconvex functions is given by the class of pseudoconvex functions. For $n = 1$, Pini (1991) has proved that pseudoconvex functions coincide with invex functions. For an example of an invex function which is not pseudoconvex, see the previously considered function

$$f(x, y) = y(x^2 - 1)^2$$

defined on $C = \{(x, y) \in \mathbb{R}^2 : y > 0\}$. Another example is given by Ben-Israel and Mond (1986), where there is a misprint. The function to be considered is

$$f(x_1, x_2) = x_1^3 + x_1 - 10x_2^3 - x_2.$$

Since there are no stationary points, f is invex. Taking $u = (0, 0)$, $x_1 = 2$, $x_2 = 1$, gives $f(x) - f(u) < 0$, but $(x - u)^\top \nabla f(u) > 0$, so that f is not quasiconvex and hence not pseudoconvex.

As previously said, invex and related functions have attracted many authors who have produced on these subjects several papers. The "critical" papers of Cambini and Carosi (2019), Caprari (2003), Martinez-Legaz (2009), Zalinescu (2014) and others, have put into evidence several flaws appeared in various papers regarding the above subjects. The list of wrong or doubtful results is too long... We point out only a curious statement of T. Antczak (2005), who introduces the class of *r-invex functions;* at page 560 of the quoted paper, the author states: "As (it) is known, invex functions are characterized by the fact that each stationary point of a function is its global minimum point. It turns out that the same result is true for a *wider class* (italics ours) of generalized convex functions, that is for r-invex functions". How its is possible that two functional classes, having the same characterization, are such that one of them is wider than the other one? This is for me a mistery.

Invex functions have been largely used also in obtaining various optimality and duality conditions in vector optimization problems. The related literature is quite vaste. We take into consideration only the papers of Osuna-Gomez, Rufian-Lizana and Ruiz-Canales (1998) and Osuna-Gomez, Beato-Moreno and

Rufian-Lizana (1999). In the said papers it is stated that "every vector Kuhn-Tucker point is weakly efficient if and only if this problem is KT-pseudoinvex". The statement seems to be correct but the "converse" part of the related proof is not clear and satisfactory. I rewrite this part of the proof. With reference to the 1999 paper in JMAA, let us suppose that every KT point is weakly efficient. We prove that the contrapositive implication of Definition 3.4 is satisfied. Suppose that system (13) in the proof of Osuna-Gomez and others does not have a solution. By Motzkin's theorem of the alternative, equation (12) of the same paper is satisfied. Therefore $\bar{x}$ is a WEP, which implies that system (14) has no solution. The other part of the proof is obvious. We have to point out that the above quoted paper contains other errors.

Finally, we spend some words on a paper of Martin (1985). This author defines *KT-invexity* (i. e. Kuhn-Tucker invexity), with reference to the following problem

$$(P): \quad \begin{cases} \min f(x) \\ \text{subject to: } g_i(x) \leqq 0, \ i = 1, ..., m, \end{cases}$$

where the functions are differentiable on an open set $X \subset \mathbb{R}^n$. We denote by K the *feasible set* of (P) and by $I(x^0)$, $x^0 \in K$, the set

$$I(x^0) = \left\{ i : g_i(x^0) = 0 \right\}.$$

The Kuhn-Tucker conditions at $x^0 \in K$ are

$$\nabla f(x^0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^0) = 0;$$

$$\lambda_i g_i(x^0) = 0, \quad i = 1, ..., m;$$

$$\lambda_i \geqq 0, \ i = 1, ..., m.$$

Definition 7. Problem (P) is *KT-invex* if there exists $\eta : X \times X \longrightarrow \mathbb{R}^n$ such that, for every $x, x^0 \in K$, it holds

$$f(x) - f(x^0) \geqq \eta(x, x^0)^\top \nabla f(x^0) \geqq 0,$$

$$-\eta(x, x^0)^\top \nabla g_i(x^0) \geqq 0, \ \forall i \in I(x^0).$$

Martin (1985) proves the following result.

Theorem 12. Every point x^0 which satisfies the above Kuhn-Tucker conditions for problem (P) is a global minimizer for problem (P) if and only if (P) is KT-invex.

However, there is an open question: KT-invexity is trivially satisfied by every solution x^0 of (P), by choosing $\eta(x, x^0) \equiv 0$. This is a tautological condition, as it is coincident with the definition of a solution of (P). Hanson and

Mond (1976b) study the problem of finding necessary and sufficient conditions of invex type that are not trivial, i. e. with $\eta(x, x^0)$ not identically zero for each feasible x, x^0. These authors introduce *Type I invexity,* a pointwise notion of invexity for (P), we will denote by *HM-invexity.*

Definition 8. Problem (P) is *HM-invex* at $x^0 \in K$ if there exists $\eta : X \longrightarrow \mathbb{R}^n$ such that for every $x \in K$, it holds

$$f(x) - f(x^0) - \eta(x)^\top \nabla f(x^0) \geqq 0$$

$$-\eta(x)^\top \nabla g_i(x^0) \geqq 0, \ \forall i \in I(x^0).$$

Theorem 13 (Hanson and Mond (1987b)). Let (P) be HM-invex at $x^0 \in K$. Then x^0 solves (P) if and only if x^0 satisfies the Kuhn-Tucker conditions for (P).

Furthermore, Hanson and Mond (1987b) prove that, under an additional condition on the contraints of (P), a vector-valued function $\eta(x)$ must exist, which is *not identically zero.*

Theorem 14. Let $x^0 \in K$ be a Kuhn-Tucker point for (P) and let be $card(I(x^0)) < n$. Then x^0 solves (P) if and only if (P) is HM-invex with respect to the vector-valued function $\eta(x)$, which is not identically zero, for each $x \in K$.

Remark 4. The condition $card(I(x^0)) < n$ cannot in general be relaxed, as shown by the following example, where $m = n = 1$.

$$\min x, \ \text{subject to: } 1 - x \leqq 0.$$

The point $x^0 = 1$ is a Kuhn-Tucker point which solves the above problem, but if HM-invexity is imposed at $x^0 = 1$, it results $\eta(x) = 0$.

Remark 5. Invexity has been used also to state suitable constraint qualifications for problems of the type (P). See, e. g., Giorgi and Guerraggio (1998a). In particular, Hanson (1999) states that "constraint qualification requirements of the Kuhn-Tucker theory appear inherently through invexity". In order to obtain his results, the same author assumes that the Kuhn-Tucker multipliers vector is bounded. We have to remark that a classical result of Gauvin (1977) states that the set of Kuhn-Tucker multipliers satisfying the Kuhn-Tucker conditions for (P) at $x^0 \in K$, is a nonempty bounded convex set if and only if the *Mangasarian-Fromovitz constraint qualification* holds at x^0.

Remark 6. Generalized convexity has been introduced by several authors also in order to generalize the second-order characterization of convex functions. There are many papers and definitions, sometimes of doubtful meaning. On this subject, i. e. second-order generalized convexity, the paper of Zalinescu (2016) takes stock of the situation.

5. Conclusions

Generalized convexity is a fascinating subject which has given rise to several interesting theoretical results and applications. The class of generalized convex functions is by now very wide, perhaps too wide, involving several "baroque" definitions with doubtful meaning and no utility towards applications. See, e. g., the surveys of Kanniappan and Pandian (1996) and Pini and Singh (1997). We conclude the present paper by quoting a sentence of F. Giannessi (2005): "In the last three decades there has been an impressive growth of definitions of generalized convexity, both for sets and functions. The way of obtaining them is very simple: if we remove one of the many properties enjoyed by convexity or we extend one of the terms of the definition, then we obtain a generalized concept; now the same can be done with the concept just obtained, and so on in a practically endless process. Some of such generalizations are of fundamental importance; unfortunately many generalizations look like mere formal mathematics without any motivation and contribute to drive mathematics away from the real world."

References

Q. H. ANSARI, S. SCHAIBLE and J.-C. YAO (1999), *η-pseudolinearity,* Rivista di Matematica per le Scienze Economiche e Sociali, 22, 31-39.

T. ANTCZAC (2005), *r-preinvexity and r-invexity in mathematical programming,* Computers and Mathematics with Applications, 50, 551-566.

T. M. APOSTOL (1974), Mathematical Analysis, Addison-Wesley, Reading, Mass.

K. J. ARROW and A. C. ENTHOVEN (1961), *Quasiconcave programming,* Econometrica, 29, 779-800.

M. AVRIEL (1972), *r-convex functions,* Math. Programming, 2, 309-323.

M. AVRIEL, W. E. DIEWERT, S, SCHAIBLE and I. ZANG (1988), Generalized Concavity, Plenum Press, New York.

M. AVRIEL and I. ZANG (1974), *Generalized convex functions with applications to nonlinear programming;* in P. Van Moeseke (Ed.), Mathematical Programs for Activity Analysis, North Holland, Amsterdam, 23-33.

C. R. BECTOR, S. CHANDRA and M. K. BECTOR (1988), *Sufficient optimality conditions and duality for a quasiconvex programming problem,* J. Optim. Theory Appl., 59, 209-221.

C. R. BECTOR and C. SINGH (1991), *B-vex functions,* J. Optim. Theory Appl., 71, 237-253.

C. R. BECTOR, S. K. SUNEJA and C. S. LALITHA (1993), *Generalized B-vex functions and generalized B-vex programming,* J. Optim. Theory Appl., 76, 561-576.

A. BEN-ISRAEL and B. MOND (1986), *What is invexity?,* J. Austral. Math. Soc., Ser. B, 28, 1-9.

J. V. BURKE and M. C. FERRIS (1991), *Characterization of solution sets of convex programs,* Operations Research Letters, 10, 57-60.

A. CAMBINI and L. MARTEIN (2009), Generalized Convexity in Optimization, Springer Verlag, Berlin.

R. CAMBINI and L. CAROSI (2019), *A note on scalar "generalized" invexity,* J. of Information and Optimization Science, 40, 615-632.

R. CAMBINI and L. MARTEIN (2001), *First and second order characterizations of a clas of pseudoconcave vector functions;* in N. Hadjisavvas, J. E. Martinez-Legaz and J.-P. Penot (Eds.), Generalized Convexity and Generalized Monotonicity - Proceedings of the 6th International Symposium on Generalized Convexity/Monotonicity, Samos, Septermber 1999, Springer, Berlin, 144-158.

E. CAPRARI (2003), *ρ-invex functions and (F, ρ)-convex functions: properties and equivalences,* Optimization, 52, 65-74.

M. CASTELLANI and M. GIULI (2012), *A characterization of the solution set of pseudoconvex extremum problems,* J. of Convex Analysis, 19, 113-123.

B. D. CRAVEN and B. M. GLOVER (1985), *Invex functions and duality,* Bull. of the Austral. Math. Soc., Series A, 39, 1-20.

J. A. FERLAND (1972), *Mathematical programming problems with quasiconvex objective functions,* Mathematical Programming, 3, 296-301.

J. GAUVIN (1977), *A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming,* Math. Programming, 12, 136-138.

F. GIANNESSI (2005), Constrained Optimization and Image Space Analysis. Volume 1: Separation of Sets and Optimality Conditions, Springer, New York.

G. GIORGI (1984), *Quasi-convex programming revisited,* Calcolo, XXI, n, 4, 307-316.

G. GIORGI (1990), *A note on the relationships between convexity and invexity,* J. Austral. Math. Soc., Ser. B, 32, 97-99.

G. GIORGI (1994), *On sufficient optimality conditions for a quasiconvex programming problem,* J. Optim. Theory Appl., 81, 401-405.

G. GIORGI (2008a), *Some remarks on preinvex functions and other generalized convex functions,* Mathematical Reports, Romanian Academy, 10 (60), 317-325.

G. GIORGI (2008b), *On some generalizations of preinvex functions,* International Journal of Pure and Applied Mathematics, 49, 121-129.

G. GIORGI and A. GUERRAGGIO (1996), *Various types of nonsmooth invex functions,* J. of Information and Optimization Sciences, 17, 137-150.

G. GIORGI and A. GUERRAGGIO (1998a), *Constraint qualifications in the invex case,* J. of Information and Optimization Sciences, 19, 373-384.

G. GIORGI and A. GUERRAGGIO (1998b), *The notion of invexity in vector optimization: smooth and nonsmooth case;* in J.-P. Crouzeix et al. (Eds.), Generalized Convexity, Generalized Monotonicity, Kluwer Acad. Publishers, Dordrecht, 389-405.

G. GIORGI and A. GUERRAGGIO (2000), *Nonsmooth vector-valued invex functions and applications,* J. of Information and Optimization Sciences, 21, 243-255.

G. GIORGI, B. JIMÉNEZ and V. NOVO (2009), *Some applications of invexity and genealized invexity to Pareto optimization problems,* International Journal of Optimization: Theory, Methods and Applications, 1, 1-14.

G. GIORGI, B. JIMÉNEZ and V. NOVO (2023), Basic Mathematical Programming Theory, Springer, Berlin.

G. GIORGI and S. MITITELU (1983), *Extremum conditions in quasiconvex programming,* Paper A-102, Dipartimento di Ricerca Operativa e Scienze Statistiche, Università di Pisa.

G. GIORGI and E. MOLHO (1992), *Generalized invexity: relationships with generalized convexity and applications to optimality and duality conditions;* in P. Mazzoleni (Ed.), Generalized Concavity for Economic Applications, A.M.A.S.E.S., University of Pisa, Faculty of Economics, 53-70.

G. GIORGI and G. N. RUEDA (2009), *η-pseudolinearity and efficiency,* International Journal of Optimization: Theory, Methods and Applications, 1(2), 155-159.

M. GUIGNARD (1969), *Generalized Kuhn-Tucker conditions for mathematical programming problems in a Banach space,* SIAM J. Control, 7, 232-241.

N. HADJISAVVAS, S. KOMLOSI and S. SCHAIBLE (Eds.), Handbook of Generalized Convexity and Generalized Monotonicity, Springer, New York.

M. A. HANSON (1981), *On sufficiency of the Kuhn-Tucker conditions,* J. Math. Anal. Appl., 80, 545-550.

M. A. HANSON (1999), *Invexity and the Kuhn-Tucker theorem,* J. Math. Anal. Appl., 236, 594-604.

M. A. HANSON and B. MOND (1982), *Further generalizations of convexity in mathematical programming,* J. of Information and Optimization Sciences, 3, 25-32.

M. A. HANSON and B. MOND (1987a), *Convex transformable programming problems and invexity,* J. of Information and Optimization Sciences, 8, 201-207.

M. A. HANSON and B. MOND (1987b), *Necessary and sufficient conditions in constrained optimization,* Math. Programming, 37, 51-58.

G. HEAL (1984), *Equivalence of saddle-points and optima for non-concave programmes,* Advances in Appl. Math., 5, 398-415.

M. R. HESTENES (1975), Optimization Theory. The Finite-Dimensional Case, J. Wiley, New York.

R. HORST (1971), *Über mittelbar konvexe Optimierung,* Doctoral Dissertation, Technische Hochschule Darmstadt, Darmstadt.

R. HORST (1972), *Mittelbar konvexe Funktionen und Optimierungsaufgaben,* Operations Research Verfahren, XII, Verlag Anton Hain, Meisenheim am Glan, 213-228.

R. HORST (1984), *On the convexification of nonlinear programming problems: an applications-oriented survey,* European Journal of Operational Research, 15, 382-392.

V. I. IVANOV (2013a), *Characterizations of pseudoconvex functions and semistrictly quasiconvex ones,* J. Glob. Optim., 57, 677-693.

V. I. IVANOV (2013b), *Optimality conditions and characterizations of the solution sets in generalized convex problems and variational inequalities,* J. Optim. Theory Appl., 158, 65-84.

V. I. IVANOV (2019), *Characterizations of solution sets of differentiable quasiconvex programming problems,* J. Optim. Theory Appl., 181, 144-162.

V. JEYAKUMAR and X. Q. YANG (1995), *On characterizing the solution sets of pseudolinear programs,* J. Optim. Theory Appl., 87, 747-755.

P. KANNIAPPAN and P. PANDIAN (1996), *On generalized convex functions in optimization theory. A survey,* Opsearch, 33, 174-185.

R. N. KAUL and S. KAUR (1985), *Optimality criteria in nonlinear programming involving nonconvex functions,* J. Math. Anal. Appl. 105, 104-112.

S. KETABCHI and E. ANSARI-PIRI (2007), *On the solution set of convex problems and its numerical application,* J. of Computational and Applied Mathematics, 206, 288-292.

K. O. KORTANEK and I. P. EVANS (1967), *Pseudo-concave programming and Lagrange regularity,* Operations Rsearch, 15, 882-891.

H. P. KÜNZI, W. KRELLE and W. OETTLI (1966), Nonlinear Prigramming, Blaisdell Publishing Co., Waltham, Mass.

C. S. LALITHA and M. MEHTA (2009), *Characterization of solution sets of mathematical programs in terms of Lagrange multipliers,* Optimization, 58, 995-1007.

X. F. LI, J. L. DONG and Q. H. LIU (1997), *Lipschitz B-vex functions and nonsmooth programming,* J. Optim. Theory Appl., 93, 557-574.

P. O. LINDBERG (1981), *Power convex functions;* in S. Schaible and W. T. Ziemba (Eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, 153-165.

O. L. MANGASARIAN (1988), *A simple characterization of solution sets of convex programs,* Operations Research Letters, 7, 21-26.

D. H. MARTIN (1985), *The essence of invexity,* J. Optim. Theory Appl., 47, 65-76.

J. E. MARTINEZ-LEGAZ (2009), *What is invexity with respect to the same* η?, Taiwanese J. of Mathematics, 13 (2B), 753-755.

B. MARTOS (1975), Nonlinear Programming: Theory and Methods, North Holland, Amsterdam.

S. K. MISHRA and G. GIORGI (2008), Invexity and Optimization, Springer Verlag, Berlin.

S. MITITELU (1974), *Extremum conditions for nonlinear functions on convex sets,* R.A.I.R.O., 8, V-2, 113-118.

S. MITITELU (1987), *Optimality, minimax and duality in nonlinear programming,* Optimization, 18, 501-506.

P. V. MOESEKE (1974), *Saddle point in homogeneous programming without Slater condition,* Econometrica, 42, 593-596.

B. MOND (1983), *Generalized convexity in mathematical programming,* Bull. Austral. Math. Soc., 27, 185-202.

A. MUKHERJI (1989), *Quasi-concave optimization. Sufficient conditions for a maximum,* Economic Letters, 30, 341-343.

R. OSUNA-GOMEZ, A. BEATO-MORENO and A. RUFIAN-LIZANA (1999), *Generalized convexity in multiobjective programming,* J. Math. Anal. Appl., 233, 205-220.

R. OSUNA-GOMEZ, A. RUFIAN-LIZANA and P. RUIZ-CANALES (1988), *Invex functions and genearlized convexity in multiobjective programming,* J. Optim. Theory Appl., 98, 651-661.

H. X. PHU (2004a), *Is invexity weaker than convexity?,* Vietnam Journal of Mathematics, 32, 87-94.

H. X. PHU (2004b), *On some badly-solved problems with invexity,* Acta Mathematica Vietnamica, 29, 89-106.

H. X. PHU (2004c), *On a necessary optimality condition with invexity,* Acta Mathematica Vietnamica, 29, 141-148.

R. PINI (1991), *Invexity and generalized convexity,* Optimization, 22, 513-525.

R. PINI and C. SINGH (1997), *A survey of recent* $[1985-1995]$ *advances in generalized convexity with applications to duality theory and optimality conditions,* Optimization, 39, 311-360.

L. QI (2001), *On an extended Lagrange claim,* J. Optim. Theory Appl., 108, 685-688.

N. G. RUEDA (1989), *Generalized convexity in nonlinear programming,* J. Inform. Optim. Sciences, 10, 395-400.

N. G. RUEDA, C. SINGH and C. R. BECTOR (1992), *Further properties of b-vex functions,* J. Inform. Optim. Sci., 13, 195-206.

S. SCHAIBLE and W. T. ZIEMBA (Eds.) (1981), Generalized Concavity in Optimization and Economics, Academic Press, New York.

T. Q. SON and D. S. KIM (2014), *A new approach to characterize the solution set of a pseudoconvex programming problem,* J. of Computational and Applied Mathematics, 261, 333-340.

W. A. THOMPSON and D. W. PARKE (1973), *Some properties of generalized concave functions,* Operations Research, 21, 305-313.

X. M. YANG (2009), *On characterizing the solution sets of pseudoinvex extremum problems,* J. Optim. Theory Appl., 140, 537-542.

Z. WU (2008), *The convexity of the solution set of a pseudoconvex inequality,* Nonlinear Analysis, 69, 1666-1674.

Z. L. WU and S. Y. WU (2006), *Characterizations of the solution sets of convex programs and variational inequality problems,* J. Optim. Theory Appl., 130, 339-358.

C. ZALINESCU (2014), *A critical view on invexity,* J. Optim. Theory Appl., 162, 695-704.

C. ZALINESCU (2016), *On second-order generalized convexity,* J. Optim. Theory Appl., 168, 802-829.

I. ZANG (1981), *Concavifiability of* C^2*-functions; a unified exposition;* in S. Schaible and W. T. Ziemba (Eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, 131-152.