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Giorgio Giorgi
(Università degli Studi di Pavia)

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Via San Felice, 5
I-27100 Pavia

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Some Investigations on Saddle Points of the Lagrangian Function

Giorgio Giorgi (*)

(*) Department of Economics and Management - University of Pavia - Via S. Felice, 5 - 27100 Pavia (Italy). e-mail: giorgio.giorgi@unipv.it

Abstract. We take into consideration the classical saddle points conditions of the Lagrangian function for a convex (or concave) minimization problem (maximization problem). After recalling some basic results, we give a survey of several constraint qualifications and regularity conditions assuring that the Kuhn-Tucker saddle points conditions hold. Then we discuss the existence of saddle points conditions for some classes of functions more general than the class of convex (or concave) functions. Finally, we discuss saddle points conditions for programming problems with convex (or concave) constraints, linear affine constraints and a set constraint.

Key words: Saddle points conditions, convex programming, concave programming, constraint qualifications, regularity conditions.

AMS subject classifications: 90C30, 90C25.

1. Introduction

A classical approach in obtaining optimality conditions for nonlinear programming problems, in absence of differentiability, is given by the *saddle points conditions* of the Lagrangian function of the problem. The pioneering paper of Kuhn and Tucker on nonlinear programming of 1951 just starts from this approach and then, by adding a differentiability assumption on the functions involved in the problem, it is shown the link between saddle points of the Lagrangian function and the Kuhn-Tucker (or better, the Karush-Kuhn-Tucker) conditions. The present paper is concerned with some questions related to this classical approach, questions we think it may be useful to describe and precise. The paper is organized as follows. Section 2 is concerned with some basic notions on saddle points and on their connections with dual problems and min-max problems. Section 3 is concerned with constraint qualifications for saddle points conditions and with regularity conditions, i. e. with conditions involving also the objective function. It is well known that in the context of nonlinear programming theory, the characterization of optimal points in terms of saddle points of the Lagrangian function is dependent upon the convexity (or the concavity) of the function involved in the problem. Section 4 is concerned with some classes of functions, besides the convex case, for which the saddle points conditions continue to hold. Section 5 discusses saddle points conditions for problems with convex (or convave) constraints, linear affine constraints and a set constraint.

We point out that there are several other questions concerned with saddle points of the Lagrangian function, questions that will not be treated in the present paper. Besides algorithmic procedures, the usual convexity (or concavity) assumptions on the functions involved in the problem could be relaxed via a *modified* (or *augmented*) *Lagrangian function.* We point out that in several papers, only local optimality conditions are obtained in the above approach and that differentiability assumptions are often imposed. We quote only the works of Arrow, Gould and Howe (1973), Bertsekas (1982), Cambini (1986), Giannessi (1980, 1984), Gould (1972), Golshtein and Tretyakov (1979), Li (1997), Li and Sun (2001), Li, Wang, Liang and Pardalos (2007), Rockafellar (1974, 1993), Smith and Vanderlinde (1972), Xu (1997).

The classical treatment of saddle points for nonlinear programming problems can be seen in several books on Convex Analysis and Nonlinear Programming, such as, e. g., Bazaraa and Shetty (1976), Bazaraa, Sherali and Shetty (1973), Bertsekas, Nedic and Ozdaglar (2003), Giorgi, Guerraggio and Thierfelder (2004), Giorgi, Jiménez and Novo (2023), Güler (2010), Mangasarian (1969), Martos (1975), Rockafellar (1970), Stoer and Witzgall (1970).

2. Some basic notions

Let us consider the following nonlinear programming problem:

$$(P): \qquad \begin{cases} \min f(x) \\ \text{subject to: } g_i(x) \leqq 0, \ i = 1, ..., m, \\ x \in X, \end{cases}$$

where $X \subset \mathbb{R}^n$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., m$. If X is a convex set, f and every g_i, $i = 1, ..., m$, are *convex* on X, then (P) is a so-called "convex minimization problem". Consider the problem

$$(P_1): \qquad \begin{cases} \max f(x) \\ \text{subject to: } g_i(x) \geqq 0, \ i = 1, ..., m, \\ x \in X, \end{cases}$$

where $X \subset \mathbb{R}^n$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and every $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., m$. If X is a convex set, f and every g_i, $i = 1, ..., m$, are *concave* on X, then (P_1) is a so-called "concave maximization problem". The *Lagrangian function*

$$\mathfrak{L}(x, \lambda) = f(x) + \lambda^\top g(x), \quad \lambda \geqq 0 \ (\lambda \in \mathbb{R}^m)$$

is therefore the same for both problems. The pair (x^0, λ^0), $x^0 \in X$, $\lambda^0 \geqq 0$, is a *(Kuhn-Tucker) saddle point* for (P) if

$$\mathfrak{L}(x^0, \lambda) \leqq \mathfrak{L}(x^0, \lambda^0) \leqq \mathfrak{L}(x, \lambda^0), \quad \forall x \in X, \ \forall \lambda \geqq 0. \tag{1}$$

The pair (x^0, λ^0), $x^0 \in X$, $\lambda^0 \geqq 0$, is a *(Kuhn-Tucker) saddle point* for (P_1) if

$$\mathfrak{L}(x, \lambda^0) \leqq \mathfrak{L}(x^0, \lambda^0) \leqq \mathfrak{L}(x^0, \lambda), \quad \forall x \in X, \ \forall \lambda \geqq 0.$$

In what follows we shall be mainly concerned with (P) and with relation (1), called also *saddle points conditions.* If we consider the *Lagrange-Fritz John function,* defined as follows

$$\mathfrak{L}(x, \alpha_0, u) = \alpha_0 f(x) + u^\top g(x),$$

with $(\alpha_0, u) \geqq 0$, $(\alpha_0, u) \neq 0$, $(\alpha_0, u) \in \mathbb{R}^{m+1}$, we have the following *Fritz John saddle points conditions:*

$$\mathfrak{L}(x^0, \alpha_0, u) \leqq \mathfrak{L}(x^0, \alpha_0, u^0) \leqq \mathfrak{L}(x, \alpha_0, u^0), \quad \forall x \in X, \ \forall u \geqq 0. \tag{2}$$

Obviously, if (x^0, α_0, u^0) is a solution of (2) and $\alpha_0 > 0$, then $(x^0, u^0/\alpha_0)$ is a solution of (1) and if (x^0, λ^0) is a solution of (1), then $(x^0, 1, \lambda^0)$ is a solution of (2). More generally, with reference to (1), we have the following definition.

Definition 1. Let $\phi(x, y)$ be a real-valued function defined on the Cartesian product $X \times Y$, where $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ and $x \in X$, $y \in Y$. A pair $(x^0, y^0) \in X \times Y$ is called a *saddle point* of $\phi(x, y)$ if

$$\phi(x^0, y) \leqq \phi(x^0, y^0) \leqq \phi(x, y^0), \quad \forall x \in X, \ \forall y \in Y.$$

That is, (x^0, y^0) is a saddle point of ϕ if x^0 minimizes $\phi(\cdot, y^0)$ over X and y^0 maximizes $\phi(x^0, \cdot)$ over Y. Clearly, a saddle point may never exist and even if it exists, it is not necessarily unique. Saddle points of a function have the following nice interchangeability property.

• Let $\phi : X \times Y \longrightarrow \mathbb{R}$ and let (x^1, y^1) and (x^2, y^2) be saddle points of ϕ. Then (x^1, y^2) and (x^2, y^1) are also saddle points and, moreover,

$$\phi(x^1, y^1) = \phi(x^2, y^1) = \phi(x^1, y^2) = \phi(x^2, y^2).$$

In a *min-max problem*, given $\phi : X \times Y \longrightarrow \mathbb{R}$, with $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, the following problems are considered:

$$\sup_{y \in Y} \inf_{x \in X} \phi(x, y);$$
$$\inf_{x \in X} \sup_{y \in Y} \phi(x, y).$$

In general these two quantities may not be equal, although it is always true that

$$\sup_{y \in Y} \inf_{x \in X} \phi(x, y) \leqq \inf_{x \in X} \sup_{y \in Y} \phi(x, y).$$

This is called *min-max inequality.* If the equality holds true, we speak of *min-max equality.* It is possible to prove the following proposition.

Theorem 1. The pair (x^0, y^0) is a saddle point of ϕ, that is

$$\phi(x^0, y) \leqq \phi(x^0, y^0) \leqq \phi(x, y^0), \quad \forall x \in X, \ \forall y \in Y,$$

if and only if the min-max equality holds and

$$x^0 \in \arg\min_{x \in X} \sup_{y \in Y} \phi(x, y); \quad y^0 \in \arg\max_{y \in Y} \inf_{x \in X} \phi(x, y).$$

For other considerations on min-max problems see, e. g., Karamardian (1967), Mangasarian and Ponstein (1965), Stoer (1963). Now we come back to problem (P). We have the following basic result.

Theorem 2. Consider problem (P). Let $X \subset \mathbb{R}^n$ be a convex set, f and every g_i, $i = 1, ..., m$, convex functions on X and let x^0 be a solution of the problem. Then there exist multipliers $\alpha_0 \geqq 0$, $u_1 \geqq 0, ..., u_m \geqq 0$, *not all zero,* such that the Fritz John saddle points conditions (2) hold. Moreover, we have ("complementary slackness condition") $(u^0)^\top g(x^0) = 0$.

The above theorem is essentially due to Uzawa (1958), even if it appears in a modified form also in the pioneering paper of 1951 of Kuhn and Tucker, where, however, differentiability is assumed. The best way to prove Theorem 2 is to use the following result of Fan, Glicksberg and Hoffman (1957), which is a generalization to the nonlinear case of the *Gordan theorem of the alternative.*

Theorem 3 (Fan-Glicksberg-Hoffman). Let $\varphi_1(x)$, $\varphi_2(x), ..., \varphi_m(x)$ be real-valued convex functions on the convex set $X \subset \mathbb{R}^n$. Then the system

$$\begin{cases} \varphi_1(x) < 0 \\ \varphi_2(x) < 0 \\ \quad \vdots \\ \varphi_m(x) < 0 \end{cases}$$

has no solution $x \in X$ if and only if there exists a vector $p \in \mathbb{R}^m$, $p \geqq 0$, *but not zero,* such that

$$p_1\varphi_1(x) + p_2\varphi_2(x) + ... + p_m\varphi_m(x) = p^\top \varphi(x) \geqq 0, \ \forall x \in X.$$

Note that if $\varphi_1(x)$, $\varphi_2(x), ..., \varphi_m(x)$ are *linear functions,* we obtain the theorem of the alternative of Gordan: either the system

$$Ax < 0,$$

with A matrix of order (m, n), $x \in \mathbb{R}^n$, has a solution or the system

$$p^\top A = 0, \ p \geqq 0, \ p \neq 0,$$

has a solution, but never both.

In order to obtain the thesis of Theorem 2 it is sufficient to define $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^{m+1}$ as follows

$$\varphi(x) = \begin{bmatrix} f(x) - f(x^0) \\ g(x) \end{bmatrix},$$

function which is convex on X. In this way, by means of Theorem 3, we obtain the second inequality of (2), as $(u^0)^\top g(x^0) \leqq 0$, and from the inequality

$$\alpha_0(f(x) - f(x^0)) + (u^0)^\top g(x^0) \geqq 0, \ \forall x \in X,$$

we have, with $x = x^0$, $(u^0)^\top g(x^0) \geqq 0$. So we obtain $(u^0)^\top g(x^0) = 0$ and the second inequality of (2). The first inequality is trivial. □

It is well known that, without an appropriate *constraint qualification,* we can have in (2) $\alpha_0 = 0$. One of the most used constraint qualifications for this kind of problems is the *Slater constraint qualification*

$$(S): \text{ Let every } g_i(x), \ i = 1, ..., m, \text{ be convex on the convex set } X \subset \mathbb{R}^n$$

$$\text{and let exist a vector } \bar{x} \in X \text{ such that } g(\bar{x}) < 0.$$

Theorem 4. Let the assumptions of Theorem 3 be satisfied and let (S) be satisfied. Then there exists a pair (x^0, λ^0) such that the saddle points conditions (1) hold.

Proof. Assume, absurdly, that in (2) $\alpha_0 = 0$. But then, being $u^0 \geqq 0$ and $u^0 \neq 0$, we have

$$\sum_{i=1}^{m} u_i^0 g_i(x) \geqq 0, \ \forall x \in X,$$

inequality which holds also for $x = \bar{x}$, in contradiction with the validity of (S). □

It is well known that if conditions (1) hold, then x^0 is a solution of (P), without assuming any convexity property. We note that, being $u_i^0 = 0$, $\forall i \notin I(x^0)$, where

$$I(x^0) = \left\{ i : g_i(x^0) = 0 \right\},$$

thanks to the complementary slackness condition, we can also write the last inequality as

$$\sum_{i \in I(x^0)} u_i^0 g_i(x) \geqq 0, \ \forall x \in X,$$

and thus require a weaker version of the usual Slater condition:

$$(S_w): \text{ Every } g_i(x), \ i = 1, ..., m, \text{ is convex on the convex set } X \subset \mathbb{R}^n$$

and there exists $\bar{x} \in X$ such that $g_i(\bar{x}) < 0, \quad \forall i \in I(x^0)$.

Also the relations between saddle points conditions (and hence a min-max problems) with *dual problems* are well known. First we recall the following characterization of saddle points for problem (P).

Theorem 5. The pair (x^0, λ^0), $x^0 \in X \subset \mathbb{R}^n$, $\lambda^0 \in \mathbb{R}^m_+$, is a saddle point for (P) if and only if the following conditions hold:
(i) $\mathfrak{L}(x^0, \lambda^0) = \min_{x \in X} \mathfrak{L}(x, \lambda^0)$,
(ii) $g(x^0) \leqq 0$,
(iii) $(\lambda^0)^\top g(x^0) = 0$.

Then we remark that problem (P) can be reformulated as follows.

$$(P): \qquad \min_{x \in X} \sup_{\lambda \in \mathbb{R}^m_+} \mathfrak{L}(x, \lambda),$$

where, as before, $\mathfrak{L}(x, \lambda) = f(x) + \lambda^\top g(x)$ and

$$\sup_{\lambda \in \mathbb{R}^m_+} \left[f(x) + \lambda^\top g(x) \right] = \begin{cases} f(x), & \text{for } g(x) \leqq 0, \ x \in X, \\ & +\infty \text{ otherwise.} \end{cases}$$

(P) is called also the "primal problem", whereas the problem

$$(D): \qquad \max_{\lambda \in \mathbb{R}^m_+} \inf_{x \in X} \mathfrak{L}(x, \lambda)$$

is the *(Lagrangian) dual problem of* (P). We note that $\mathfrak{L}(x, \lambda)$ is, as a function of λ, a concave function and hence the function $d(\lambda) = \inf_{x \in X} \mathfrak{L}(x, \lambda)$ is a concave function. We have the following basic result.

Theorem 6. The following propositions are equivalent:
(i) The Lagrangian function $\mathfrak{L}(x, \lambda)$ has a saddle point $(x^0, \lambda^0) \in X \times \mathbb{R}^m_+$.
(ii) The vector x^0 is a solution of (P), the vector $\lambda^0 \in \mathbb{R}^m_+$ is a solution of (D) and the two optimal values of (P) and (D) are equal, i. e. *there is no duality gap.*

The above result is known also as the *strong duality theorem.* The *weak duality theorem* states that if x^* is a feasible point of the primal problem (P) and λ^* is a feasible point of the dual problem (D), then $f(x^*) \geqq d(\lambda^*)$, where $d(\lambda) = \inf_{x \in X} \mathfrak{L}(x, \lambda)$.

3. Constraint qualifications and regularity conditions

Any condition which ensures that in Theorem 2 relation (2) holds with $\alpha_0 > 0$ is referred in the literature as a *constraint qualification.* We have previously described the *Slater constraint qualification.* For a survey of constraint qualifications in the differentiable case, the reader may consult, e. g., Giorgi, Guerraggio and Thierfelder (2004), Giorgi, Jiménez and Novo (2023), Peterson (1973). Usually, those conditions which ensure that in Theorem 2 relation (2) holds with $\alpha_0 > 0$, but which involve also the objective function $f(x)$, are known as *regularity conditions.* Other known constraint qualifications for (P) and with reference to (2), are the following ones (see, e. g., Mangasarian (1969)).

• *Karlin constraint qualification:* let $X \subset \mathbb{R}^n$ be a convex set and let every $g_i(x)$, $i = 1, ..., m$, be convex on X. Then there exists no $p \in \mathbb{R}^m$, $p \geqq 0$, $p \neq 0$, such that $p^\top g(x) \geqq 0$, $\forall x \in X$.

• *Strict constraint qualification:* let $X \subset \mathbb{R}^n$ be a convex set and let every $g_i(x)$, $i = 1, ..., m$, be convex on X. Then the feasible set

$$K = \{x : x \in X, g_i(x) \leqq 0,\ i = 1, ..., m\}$$

satisfies the strict constraint qualification if K contains at least two distinct points x^1 and x^2 such that $g(x)$ is *strictly convex* at x^1.

For the reader's convenience we recall the following result.

Theorem 7. Slater constraint qualification and Karlin constraint qualification are equivalent. Strict constraint qualification implies Slater constraint qualification (and hence Karlin constraint qualification).

Proof. The equivalence between Slater c. q. and Karlin c. q. is immediate from Theorem 3 (Fan-Glicksberg-Hoffman). Now we prove that strict c. q. implies Slater c. q.. Since X is convex, we have, for every $\lambda \in (0, 1)$,

$$(1 - \lambda)x^1 + \lambda x^2 \in X.$$

Because g is strictly convex at x^1, it follows that

$$g((1 - \lambda)x^1 + \lambda x^2) < (1 - \lambda)g(x^1) + \lambda g(x^2) \leqq 0,$$

where the last inequality follows from the fact that $g(x^1) \leqq 0$ and $g(x^2) \leqq 0$. Then the c. q. of Slater is satisfied and hence also the c. q. of Karlin. □

Now, the following questions are rather "natural". There are constraint qualifications weaker than Slater c. q.? There are constraint qualifications or regularity conditions which are *necessary and sufficient* to obtain $\alpha_0 > 0$ in relation (2) of Theorem 2? We take into consideration, in what follows, some proposals of various authors, regarding the above questions.

A) Approach of L. Martein (1985).

This author develops, with respect to problem (P), some previous results of Giannessi (1984). See also Cambini (1982, 1986), Quang and Yen (1991). Indeed, Martein takes into consideration a minimization problem with constraints of the type $g(x) \geqq 0$. We consider problem (P), where $X \subset \mathbb{R}^n$ is a nonempty convex set, $f : X \longrightarrow \mathbb{R}$ and every $g_i : X \longrightarrow \mathbb{R}$, $i = 1, ..., m$, are convex on X. Let x^0 be a (global) minimum point of (P), set $\varphi_1(x) = f(x) - f(x^0)$ and consider the inequality

$$\theta\varphi_1(x) + (\lambda^0)^\top g(x) \geqq 0, \quad \forall x \in X, \tag{3}$$

essential in the proof of Theorem 2, where $\theta \in \mathbb{R}_+$, $\lambda^0 \in \mathbb{R}^m_+$, $(\theta, \lambda^0) \neq 0$. All conditions which ensure $\theta > 0$ in (3) are referred to as "regularity conditions", if they involve both the objective function and the constraints. Now, let us introduce the function $F : X \longrightarrow \mathbb{R} \times \mathbb{R}^m$, with $F(x) \equiv [\varphi_1(x), g_1(x), ..., g_m(x)]^\top \equiv \varphi(x)$ and the sets

$\mathcal{K} = \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u = \varphi_1(x),\ v = g(x),\ x \in X\}$;

$\mathcal{H} = \{(u, v) \in \mathbb{R} \times \mathbb{R}^m : u < 0, v \leqq 0\}$;

$\mathcal{U} = \{(u, 0) \in \mathbb{R} \times \mathbb{R}^m : u \leqq 0\}$;

$\mathcal{E}(F(x)) = \{F(x)\} - cl(\mathcal{H})$. This set is called the *conic extension of* $F(x)$ with respect to the cone $-cl(\mathcal{H})$.

$\mathcal{E} = \bigcup_{x \in X} \mathcal{E}(F(x))$.

We denote by $T(\mathcal{E}, 0)$ the *Bouligand tangent cone* (or *contingent cone*) to $\mathcal{E}$ at $0 \in cl(\mathcal{E})$; we recall the definition of this cone:

$$T(S, x^0) = \left\{ \begin{array}{c} y \in \mathbb{R}^n : \exists t_n \longrightarrow 0^+,\ \exists y^n \longrightarrow y \text{ such that } x^0 + t_n y^n \in S \\ \text{for each integer } n \end{array} \right\},$$

where $S \subset \mathbb{R}^n$ and $x^0 \in cl(S)$.

Martein (1985) proves the following result.

Theorem 8. A necessary and sufficient regularity condition to have $\theta > 0$ in (3), i. e. to have a saddle point condition for problem (P), is

$$T(\mathcal{E}, 0) \cap int(\mathcal{U}) = \varnothing. \tag{4}$$

If (P) is not a convex problem, condition (4) is still necessary, even not sufficient to have $\theta > 0$ in (3).

Always with reference to the convex problem (P), another condition equivalent to (4) is expressed by the following result (see also Cambini (1982,1986)).

Theorem 9. Consider the convex problem (P); then condition (4) is equivalent to the following condition:

For every sequence $\{x^r\} \subset X$, which converges to a global minimum point of problem (P), and for every positive sequence $\{\alpha_r\} \subset \mathbb{R}$, either we have

$$\lim_{r \longrightarrow +\infty} \alpha_r(\varphi_1(x^r), g(x^r)) \neq (u_0, 0),\ \text{with } u_0 < 0,$$

or such a limit does not exist.

B) Approach of Kaplan and Rubinstein (1969).

These authors consider the *concave* problem (P_1), where f and $g_1, ..., g_m$ are defined on some open convex set $G \subset \mathbb{R}^n$, and a convex set $X^0 \subset G$. They denote by Ω the feasible set, i. e. $\Omega = \cap_{i=0}^m X^i$, where $X^i = \{x \in G : g_i(x) \geqq 0\}$, $i = 1, ..., m$. Subsequently they study conditions on the set X^0 and the functions $g_1, ..., g_m$, under which the following statement holds:

- In order for the vector $x^0 \in X^0$ to be optimal in problem (P_1) the existence of a vector y^0 such that (x^0, y^0) is a saddle point of the Lagrangian function $\mathfrak{L}(x, y)$ is not only sufficient but also necessary.

Note that in the above problem the objective function does not appear, i. e. the authors make reference to a constraint qualification for (P_1). By the *limit cone of a convex set* $X \subset \mathbb{R}^n$ *at the point* $x^0 \in X$ the authors mean the set $X(x^0) \equiv cl(K(x^0))$, where $K(x^0) = \underset{x\in X}{\cup}\Pi(x^0)$ and $\Pi(x^0) = \left\{z : z = x^0 + t(x - x^0),\ t > 0, x \in X\right\}$.

They indicate the limit cone of the set X at the point x^r, $r = 0, 1, ..., r-1$, by $X^r(x^0, x^1, ..., x^r)$. The main result of these authors is expressed in the following theorem.

Theorem 10. Set in problem (P_1) $X^j = \{x : g_j(x) \geqq 0\}$, hence the problem becomes $\max\left\{f(x) : x \in \cap_{j=0}^m X^j\right\}$. If in this problem it holds $\sup g_j(x) > 0$, $j = 1, ..., m$, then in order that the related saddle points conditions

$$\mathfrak{L}(x, \lambda^0) \leqq \mathfrak{L}(x^0, \lambda^0) \leqq \mathfrak{L}(x^0, \lambda),\ \forall x \in X^0, \forall \lambda \geqq 0,$$

hold for any concave function f having a maximum at x^0, it is necessary and sufficient that

$$\bigcap_{j=0}^{m} X^j(x^0, x^1, ..., x^r) = \left((\bigcap_{j=0}^{m} X^j), x^0, x^1, ..., x^r\right)$$

for any finite choice of the points $x^0, x^1, ..., x^r$, with $x^0 \in \cap_{j=0}^m X^j$ and $x^k \in \left((\cap_{j=0}^m X^j), x^0, x^1, ..., x^{k-1}\right)$.

C) Approach of Gale (1967) and Geoffrion (1971).

These authors consider the *convex* problem (P) and define the *perturbation function* $v(\cdot)$ associated with (P) as:

$$v(y) = \inf_{x\in X} \{f(x) : g(x) \leqq y\},$$

where $y \in \mathbb{R}^m$ is called the *perturbation vector*. The problem (P) is said to be *stable* if $v(0)$ is finite and there exists a scalar $M > 0$ such that

$$\frac{v(0) - v(y)}{\|y\|} \leqq M,\ \forall y \neq 0.$$

The following result is proved by Gale (1967) and Geoffrion (1971).

Theorem 11. Assume that (P) has an optimal solution at x^0; then there exists a vector λ^0 such that the saddle points conditions (1) hold if and only if (P) is stable.

Note that the conditions of the previous result are *regularity conditions,* not constraint qualifications.

D) Approach of P. V. Moeseke (1974).

This author considers the concave problem (P_1) and substitutes the usual Slater condition with the assumption that f and every g_i, $i = 1, ..., m$, are concave on the convex set $X \subset \mathbb{R}^n$ and *(positively) homogeneous of degree* $r \neq 0$. We recall that a function $\varphi : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, where X is a *cone* in $\mathbb{R}^n$, is positively homogeneous of degree $r \neq 0$ if

$$\varphi(\rho x) = \rho^r \varphi(x), \text{ for all } x \in X \text{ and for all } \rho > 0.$$

Hence, the class of positively homogeneous functions includes the class of homogeneous functions tout-court. The main result of this author is the following one.

Theorem 12. Let the above asssumptions on (P_1) be verified. The feasible point x^0 solves (P_1) if and only if the pair (x^0, λ^0), $\lambda^0 \geqq 0$, is a saddle point for (P_1) of the Lagrangian function $\mathfrak{L}(x, \lambda)$ on $X \times \mathbb{R}^m_+$, where X is a convex cone of $\mathbb{R}^n$.

We recall that a function $\varphi : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, where X is a cone, is said to be *linearly homogeneous* if it is homogeneous of degre one, i. e.,

$$\varphi(\rho x) = \rho \varphi(x), \text{ for all } x \in X \text{ and for all } \rho > 0.$$

The following result shows that linear homogeneity plus super-additivity produces concavity:

• Let $\varphi : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linearly homogeneous function defined on the cone X. Then φ is concave if and only if for every $x, y \in X$

$$f(x + y) \geqq f(x) + f(y).$$

Another result related to concavity and homogeneity is the following one.

• Let $\varphi : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a homogeneous function of degree $r \geqq 1$ on the convex cone X. If $\varphi(x) > 0$ for each $x \in X$, then f is quasiconcave if and only if it is concave.

See Cambini and Martein (2009). Another paper concerned with these topics is Le Blanc (1977).

E) Approach of H. Wolkowicz (1980).

This author considers the convex problem (P) and obtains several results on the "weakest" constraint qualification and on regularization techniques. The results are expressed, in general, in terms of subgradients and subdifferentials and, in a sense, parallel the results for the differentiable case of Gould and Tolle (1971) and Guignard (1969) on the weakest constraint qualification. Indeed, we have the following basic results which connect subdifferentials of convex functions with saddle points conditions. See Ruszczynski (2006).

Theorem 13. Assume that x^0 is a solution of the convex problem (P), where all functions $f, g_1, ..., g_m$ are convex on an open set $S \subset \mathbb{R}^n$ containing the feasible set, and that the Slater constraint qualification is satisfied. Then, there exists $\lambda^0 \in \mathbb{R}^m_+$ such that

$$0 \in \partial f(x^0) + \sum_{i=1}^{m} \lambda_i^0 \partial g_i(x^0) + N(X, x^0), \tag{5}$$

$$\lambda_i^0 g_i(x^0) = 0, \quad i = 1, ..., m. \tag{6}$$

Conversely, if for some feasible point x^0 of the convex problem (P), the conditions (5) and (6) are satisfied, then x^0 is a solution of (P).

Here $N(X, x^0)$ is the *normal cone* of the convex set X at $x^0 : N(X, x^0) = \left[cone(X - x^0)\right]^* \equiv \left\{v : v^\top (y - x^0) \leqq 0, \ \forall y \in X\right\}$. Observe that if $x^0 \in int(X)$, then $N(X, x^0) = \{0\}$.

Theorem 14. Consider the convex problem (P) and let the assumptions of Theorem 13 be satisfied. A point x^0 satisfies the optimality conditions (5) and (6) of Theorem 13, with Lagrange multipliers $\lambda_1^0, ..., \lambda_m^0$, if and only if (x^0, λ^0) is a saddle point of the Lagrangian function $\mathfrak{L}(x, \lambda)$.

Another paper concerned with this approach is Dutta and Lalitha (2013).

F) Approach of H. Oniki (1971).

This author, in an unpublished paper, but downloadable from the WEB, takes into consideration the concave problem (P_1) with $X = \mathbb{R}^n$, and proposes a "weakest" constraint qualification for the related saddle points conditions. He presents also a regularity condition, necessary and sufficient for the saddle points conditions. This author uses the well known fact that concave functions (and obviously also convex functions) possess all the (one-sided) directional derivatives of any order at any interior point of their domain. The constraint qualification and the regularity condition proposed by this author are expressed in terms of directional derivatives, not only of the first order but also of higher order. The necessary and sufficient regularity condition is given in Theorem 2 of Oniki (1971), whereas the "weakest" constraint qualification is given in Theorem 3 of the same paper.

G) Approach of Hoang Tuy (1974).

This author considers a generalization of the convex problem (P) and gives a "generalized constraint qualification", quite intricate, weaker than the usual constraint qualifications of Slater and Karlin (i. e. without assuming differentiability) under which the saddle points conditions hold.

H) Approach of Hollatz (1973).

This author consider the convex problem (P), with $X = \mathbb{R}^n$ and gives necessary and sufficient conditions of optimality in terms of saddle points of the Lagrangian function: these conditions are expressed by stating that certain convex sets must be separable by a non-vertical surface.

We have not examined the paper of Vilkov and Surovtsov (1980).

3. Other classes of functions for which saddle points conditions hold

In the previous sections we have recalled some basic properties on saddle points conditions of the Lagrangian function for the convex problem (P) or for the concave problem (P_1). We have also examined some constraint qualifications and regularity conditions. In the present section we examine the existence of classes of functions, more general than convex functions (or than concave functions), for which the basic optimality conditions in terms of the Lagrangian function continue to hold.

I) Pre-invex functions.

Ben-Israel and Mond (1986) and Weir and Mond (1988) consider the class of (non necessarily differentiable) *pre-invex functions.*

Definition 2. Let $S \subset \mathbb{R}^n$ and let be given a function $f : S \longrightarrow \mathbb{R}$; if there exists a function $\eta : S \times S \longrightarrow \mathbb{R}^n$ such that, for each $x, y \in S$ and $\lambda \in [0, 1]$, $y + \lambda\eta(x, y) \in S$, we have

$$f(y + \lambda\eta(x, y)) \leqq \lambda f(x) + (1 - \lambda)f(y),$$

then f is called *pre-invex.*

Clearly, if $\eta(x, y) = x - y$, then f is convex and S is a convex set. It can be shown that, as for convex functions, every local minimum of a pre-invex function is a global minimum and that nonnegative linear combinations of pre-invex functions are pre-invex. A simple example of a function which is pre-invex, but not convex is given by Weir and Mond (1988). Consider

$$f(x) = -|x|, \ x \in \mathbb{R}.$$

Then f is pre-invex, with η given by

$$\eta(,y) = \begin{cases} x-y, & \text{if } y \leqq 0 \text{ and } x \leqq 0 \\ x-y, & \text{if } y \geqq 0 \text{ and } x \geqq 0 \\ y-x, & \text{if } y > 0 \text{ and } x < 0 \\ y-x, & \text{if } y < 0 \text{ and } x > 0. \end{cases}$$

The basic theorem of the alternative, which involves pre-invex functions and which may be considered an extension of the Fan-Glicksberg-Hoffman Theorem (Theorem 3) is proved by Weir and Mond (1988).

Theorem 15. Let $S \subset \mathbb{R}^n$ and let $f : S \longrightarrow \mathbb{R}^m$ be a pre-invex function on S (i. e. every component of S is pre-invex on S with respect to the *same* η). Then either

$$f(x) < 0 \text{ has a solution } x \in S$$

or there exists $p \in \mathbb{R}^m$, $p \geqq 0, p \neq 0$, such that

$$p^\top f(x) \geqq 0, \ \forall x \in S,$$

but both alternatives are never true.

Now consider a minimization problem

$$(P): \qquad \begin{cases} \min f(x) \\ \text{subject to: } g(x) \leqq 0, \end{cases}$$

where $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$. The above problem (P) is said to satisfy the *generalized Slater constraint qualification* if g is pre-invex (with respect to η) and there exists a point $\bar{x} \in S$ such that $g(\bar{x}) < 0$. Then the following basic result can be easily proved.

Theorem 16. Consider the above problem (P) and assume that $f(x)$ is pre-invex with respect to η, that $g(x)$ is pre-invex with respect to the *same* η and that the generalized Slater constraint qualification holds. If (P) has a minimum at $x^0 \in S$, then there exists a vector $\lambda^0 \in \mathbb{R}^m$, $\lambda^0 \geqq 0$, such that the pair (x^0, λ^0) is a saddle point for the Lagrangian function $\mathfrak{L}(x, \lambda)$.

Jeyakumar (1988) generalizes further the above results for a class of nonsmooth and nonconvex problems in which the related functions are locally Lipschitz and satisfy some invex-type conditions. We recall that a real-valued function $h : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, S open, is said to be *locally Lipschitz at* x^0 if there exists a positive constant k and a neighbourhood $N(x^0)$ such that

$$|h(x) - h(y)| \leqq k \|x - y\|, \quad \forall x, y \in N(x^0).$$

A function $h : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *locally Lipschitz* if it is locally Lipschitz at each point $x^0 \in S$. However, it is important to note that the value of the Lipschitz constant k in general changes as we change the point x^0. If, however, a locally Lipschitz function has a uniform Lipschitz constant k for every point $x^0 \in S$, then the function is Lipschitz over the set S in the usual sense.

For a locally Lipschitz function $h(x)$ the *Clarke generalized directional derivative at* x^0 and the *Clarke subdifferential at* x^0 are, respectively, defined by

$$\begin{aligned} h^0(x^0, v) &= \limsup_{y \longrightarrow x^0, t \longrightarrow 0^+} \frac{h(y+tv) - h(y)}{t}; \\ \partial_C h(x^0) &= \left\{ \xi \in \mathbb{R}^n ; h^0(x^0, v) \geqq \xi^\top v, \ \forall v \in \mathbb{R}^n \right\}. \end{aligned}$$

Clarke (1983) has shown that, when the function $h(x)$ is convex, $\partial_C h(x^0)$ reduces to the usual subdifferential of Convex Analysis.

Definition 3. A locally Lipschitz function on the open set $S \subset \mathbb{R}^n$, $h : S \longrightarrow \mathbb{R}$, is called *$\rho$-invex at $x^0 \in S$* with respect to $\eta, \theta : S \times S \longrightarrow \mathbb{R}^n$, $\theta(x, x^0) \neq 0$, whenever $x \neq x^0$, if there exists a real number ρ such that for each $\xi \in \partial_C h(x^0)$

$$h(x) - h(x^0) \geqq \xi^\top \eta(x, x^0) + \rho \left\| \theta(x, x^0 \right\|^2, \ \forall x \in S. \tag{7}$$

The function $h(x)$ is called *ρ-invex* if (7) holds for each $x^0 \in S$. If $\rho > 0$, then $h(x)$ is said to be *strongly invex,* if $\rho = 0$, then $h(x)$ is said to be *invex,* if $\rho < 0$, then $h(x)$ is said to be *weakly invex.*

It is clear that

$$\text{strongly invex} \implies \text{invex} \implies \text{weakly invex}.$$

See also Caprari (2003).

Consider the minimization problem (P), with $X = \mathbb{R}^n$ and where the functions of the problem are locally Lipschitz at a feasible point x^0. For this problem a pair (x^0, λ^0), with x^0 feasible point and $\lambda^0 \in \mathbb{R}^m_+$, is said to be a *critical point* if

$$0 \in \partial_C f(x^0) + \sum_{i=1}^m \lambda_i^0 \partial_C g_i(x^0),$$

$$\lambda_i^0 g_i(x^0) = 0, \ i = 1, ..., m.$$

Clarke (1983) has shown that if x^0 is a local minimum point for the above problem, and an appropriate constraint qualification holds (e. g. the "calmness constraint qualification" of Clarke), then there exists $\lambda^0 \in \mathbb{R}^m_+$ such that (x^0, λ^0) is a critical point. The main results of Jeyakumar (1988) on saddle points condition are contained in the following theorem.

Theorem 17. For the locally Lipschitz problem (P), let the function $f(x)$ be ρ_0-invex at a feasible point x^0, and let g_i, $i = 1, ..., m$, be ρ_i-invex with respect to the *same* functions η and θ. Suppose that (x^0, λ^0) is a critical point for (P) and that $(\rho_0 + \sum_{i=1}^m \lambda_i^0 \rho_i) \geqq 0$. Then the pair (x^0, λ^0) is a saddle point of the Lagrangian function:

$$\mathfrak{L}(x^0, \lambda) \leqq \mathfrak{L}(x^0, \lambda^0) \leqq \mathfrak{L}(x, \lambda^0), \quad \forall x \in \mathbb{R}^n, \ \forall \lambda \in \mathbb{R}_+^m.$$

Conversely, if (x^0, λ^0) is a saddle point, then x^0 is a solution of (P) and the complementary slackness conditions hold, i. e. $\lambda_i^0 g_i(x^0) = 0$, $i = 1, ..., m$.

II) E-convex functions.

The definitions of *E-convex sets* and *E-convex functions* were introduced by Youness (1999), as a generalization of convex sets and convex functions. However, as shown by Yang (2001), some of the results of Youness are incorrect. We recall the basic definitions.

Definition 4.

(*i*) A set $S \subset \mathbb{R}^n$ is said to be *E-convex* if there is a map $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$(1 - \lambda)E(x) + \lambda E(y) \in S, \quad \forall \lambda \in [0, 1] \, .$$

(*ii*) A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be *E-convex* on a set $S \subset \mathbb{R}^n$ if there is a map $E : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that S is an E-convex set and

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leqq \lambda f(E(x)) + (1 - \lambda) f(E(y)), \quad \forall x, y \in S, \ \forall \lambda \in [0, 1] \, .$$

Youness (2001) considers problem (P) with $X = \mathbb{R}^n$, K the feasible set, and proves the following result.

Theorem 18. Let in the above problem (P) the function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the functions $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., m$, be real valued E-convex functions on the set $K = \{x \in \mathbb{R}^n : g_i(x) \leqq 0\}$; let $g_i(x)$, $i = 1, ..., m$, satisfy a Slater-type constraint qualification. If $E(K)$ is convex, $E(K) \subset K$ and x^0 is a solution of (P), then there is a pair (x^0, λ^0), $\lambda^0 \in \mathbb{R}_+^m$, such that (x^0, λ^0) is a saddle point of the "modified" Lagrangian function $\phi(x, \lambda) = (f \circ E)x + \lambda^\top (g \circ E)x$.

The incorrect results of Youness are pointed out also by Jian (2003). Also Chen (2002) gives some counterexamples to the claims of Youness and proposes the notion of *semi-E-convex functions,* in order to amend the said incorrect results. Also the above Theorem 18 (Theorem 2.3 in Youness (2001)) is not expounded in a clear way: for example, the Slater constraint qualification is not specified.

III) Convexifiable functions.

An interesting result on the subject of the present section is given by Heal (1984) who takes into consideration the so-called *convexifiable functions* or *convex range transformable functions.* See also Avriel and Zang (1974), Horst (1984), Mond (1983), Zang (1981).

Definition 5. Let $f : C \longrightarrow \mathbb{R}$ be defined on the convex set $C \subset \mathbb{R}^n$, and denote by $I = I_f(C)$ the range of $f(x)$, i. e. the image of C under f. Then $f(x)$ is said to be *convexifiable* or *convex range transformable* or briefly *h-convex,* if there exists a continuous strictly monotone increasing function $h : I \longrightarrow \mathbb{R}$, such that $h\left[f(x)\right]$ is convex on C, that is

$$h\left[f(\lambda x^1 + (1-\lambda)x^2\right] \leqq \lambda h\left[f(x^1)\right] + (1-\lambda)h\left[f(x^2\right],\ \forall x^1, x^2 \in C,\ \forall \lambda \in [0,1]\,,$$

that is

$$\left[f(\lambda x^1 + (1-\lambda)x^2\right] \leqq h^{-1}\left\{\lambda h\left[f(x^1)\right] + (1-\lambda)h\left[f(x^2\right]\right\},\ \forall x^1, x^2 \in C,\ \forall \lambda \in [0,1]\,.$$

Examples of convexifiable functions are *r-convex functions,* introduced by Avriel (1972) and Martos (1975), obtained by taking $h(x) = e^{rx}$, $x \in \mathbb{R}$, $r \neq 0$. Then $h^{-1}(x) = \log x^{1/r}$, $x > 0$. Therefore we have

$$f(\lambda x^1 + (1-\lambda)x^2) \leqq \begin{cases} \log\left[\lambda e^{rf(x^1)} + (1-\lambda)e^{rf(x^2)}\right]^{\frac{1}{r}},\ r \neq 0, \\ \lambda f(x^1) + (1-\lambda)f(x^2),\ r = 0, \end{cases}$$

for every $x^1, x^2 \in C$, $\forall \lambda \in [0,1]$.

Taking $h(x) = x^p$, $x > 0$, $p \neq 0$, we have the *power convex functions,* or *p-convex functions,* studied by Avriel (1972) and Lindberg (1981). We have $h^{-1}(x) = x^{1/p}$, $x > 0$, and hence

$$f((\lambda x^1 + (1-\lambda)x^2) \leqq \begin{cases} \left[\lambda f^p(x^1) + (1-\lambda)f^p(x^2)\right]^{\frac{1}{p}},\ p \neq 0, \\ f^{\lambda}(x^1) \cdot f^{(1-\lambda)}(x^2),\ p = 0. \end{cases}$$

It can be proved that any h-convex function is *semistrictly quasiconvex,* but in general the converse does not hold. Moreover, if f and h are differentiable, h-convex functions are *pseudoconvex,* but in general the converse does not hold. Some basic properties typical of convex functions continue to hold for the class of convexifiable functions; for example, every local minimum of h-convex functions is also a global one, and if f and h are differentiable, then every stationary point of f is a global minimum point over C. in particular, the set of global minimizers of h-convex functions is a convex set and every h-convex function is continuous in the interior of its domain C. Heal (1984) proves that the saddle points conditions continue to hold for the class of h-convex functions.

IV) Saddle points with a quasiconcave or a quasiconvex objective function.

Fujimoto (1978) takes into consideration problem (P_1), with $X = \mathbb{R}^n_+$ and makes the following additional assumptions:

- $f(x)$ is continuous and quasiconcave on $\mathbb{R}^n_+$.
- There exists an x in $\mathbb{R}^n_+$ such that $g(x) > 0$ ("Slater condition").

• $f(x)$ is non-decreasing and $f(x^1) < f(x^2)$ for any two vectors in $\mathbb{R}^n_+$ such that $x^1 < x^2$.

Furthermore, this author introduces the following notations

$$A(x^*) = \left\{x : f(x) \geqq f(x^*),\ x \in \mathbb{R}^n_+\right\}$$

for a given $x^* \in \mathbb{R}^n_+$.

$$B = \left\{x : g(x) \geqq 0\ ,\ x \in \mathbb{R}^n_+\right\},$$

i. e. B is the feasible set.

The author then formulates the following modified saddle-point problem:

• (MS) : Find a triplet of vectors (x^0, y^0, p^0) such that $x^0 \in \mathbb{R}^n_+$, $y^0 \in \mathbb{R}^m_+$, $p^0 \in \mathbb{R}^n_+$, $p^0 \neq 0$, $(p^0)^\top x \geqq (p^0)^\top x^0$ for $x \in A(x^0)$, $(p^0)^\top x \leqq (p^0)^\top x^0$, for $x \in B$, such that

$$\begin{aligned}(p^0)^\top x + (y^0)^\top g(x) \quad &\leqq \quad (p^0)^\top x^0 + (y^0)^\top g(x^0) \leqq (p^0)^\top x^0 + (y)^\top g(x^0),\\ \forall x \quad &\in \quad \mathbb{R}^n_+, \forall\ \ y \in\ \ \mathbb{R}^m_+.\end{aligned}$$

Fujimoto (1978) proves the following result.

Theorem 19. Given the above assumptions, x^0 is a solution of problem (P_1) if and only if there exists a pair of vectors (y^0, p^0) such that the triplet (x^0, y^0, p^0) is a solution of the problem (MS).

Another approach to treat the convex problem (P), where the objective function is quasiconvex, in terms of saddle-points conditions for a modified Lagrangian function, is given by Luenberger (1968). This author takes into consideration problem (P), with $f(x)$ quasiconvex on the convex set $X \subset \mathbb{R}^n$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ convex on X. In addition the following two "regularity conditions" are assumed to hold.

(i) The function $f(x)$ is upper semicontinuous along lines, i. e. for every $x^1, x^2 \in X$, $f(\alpha x^1 + (1-\alpha)x^2)$ is an upper semicontinuous function of α for $\alpha \in [0,1]$.

(ii) There is an $x^1 \in X$ such that $g(x^2) < 0$ (Slater c.q.).

One of the results presented by Luenberger (1968) is the following one.

Theorem 20. If x^0 solves (P), under the above assumptions, then there is a $\lambda^0 \in \mathbb{R}^m$, $\lambda^0 \geqq 0$, $\lambda^0 \neq 0$, such that x^0 solves the following problem

$$\min f(x),\ \text{subject to: } (\lambda^0)^\top g(x) \leqq 0. \tag{8}$$

The results of Theorem 20 can be rephrased as: x^0 solves (P) if and only if there exists a vector λ^0 such that (x^0, λ^0) is a saddle point of $\varphi(x, \lambda)$ on $X \times S$, where S is the unit simplex in $\mathbb{R}^m$, i. e. $S = \{\lambda : \lambda \geqq 0,\ \sum_{i=1}^m \lambda_i = 1\}$ and

$$\varphi(x, \lambda) = \begin{cases} f(x), \text{ if } \lambda^\top g(x) \leqq 0 \\ +\infty, \text{ if } \lambda^\top g(x) > 0. \end{cases}$$

Note that in general, contrary to the convex case, we have not $(\lambda^0)^\top g(x^0) = 0$ for the above quasiconvex programming problem. In order to obtain the said complementary slackness condition, Luenberger proves the following result.

Theorem 21. Let x^0 solve problem (P) and define $\mu_0 = f(x^0)$. Then, either there is an $\bar{x} \in X$, $g(\bar{x}) < 0$ and $f(\bar{x}) = \mu_0$, or there is a $\lambda^0 \geqq 0$ such that x^0 solves problem (8) and $(\lambda^0)^\top g(x^0) = 0$.

5. Saddle points conditions with linear constraints

Apparently the first author concerned with saddle points conditions for a programming problem in which, besides convex or concave inequality constraints, there are also linear (affine) equality constraints, is Uzawa (1958) in a fundamental paper. However, as pointed out by Moore (1968), Theorem 3 of Uzawa's paper is not fully correct, perhaps due to a misprint. Moore gives a counterexample to Uzawa's claim. The paper of Moore contains several interesting results on saddle points conditions for a vector optimization problem, under quite general assumptions. The scalar version of the main results of Moore is reported by Takayama (1985).

Theorem 22. Let $f(x)$ be a real-valued concave function on X, a convex subset of $\mathbb{R}^n$, and let $g : X \longrightarrow \mathbb{R}^m$, with

$$g(x) = \begin{bmatrix} g^1(x) \\ g^2(x) \end{bmatrix}.$$

Suppose that $\hat{x} \in X$ achieves a maximum of $f(x)$ subject to $g^1(x) \in K_1$ and $g^2(x) \in K_2$, where K_1 is a convex cone in $\mathbb{R}^{m_1}$ and K_2 is a convex cone in $\mathbb{R}^{m_2}$, with $m_1 + m_2 = m$. Assume the following:

(i) The function g^1 is linear affine, the function g^2 is K_2-*concave on* X (a function $g : X \longrightarrow \mathbb{R}^m$, where X is a convex subset of $\mathbb{R}^n$, is K-concave on X, with K convex cone in $\mathbb{R}^m$, if

$$g(\lambda x + (1-\lambda)y) - (\lambda g(x) + (1-\lambda)g(y)) \in K, \quad \forall x, y \in X, \ \forall \lambda \in [0,1]).$$

(ii) $int(X) \neq \varnothing$ and $int(K_2) \neq \varnothing$.

(iii) There exists $\hat{x} \in int(X)$ such that $g^1(\hat{x}) \in K_1$.

(iv) There exists $\bar{x} \in X$ such that $g^1(\bar{x}) \in K_1$ and $g^2(\bar{x}) \in int(K_2)$.

Then, there exists $\hat{\lambda} \in K^*$, where K^* denotes the nonnegative polar cone of $K = K_1 \times K_2$, such that $(\hat{x}, \hat{\lambda})$ is a saddle point of $f(x) + \lambda^\top g(x)$.

Section III of Moore's paper is quite interesting, as it is concerned with "constraint qualifications and the geometry of generalized saddle points". Lemma 6 of the said paper, here reformulated with reference to the above Theorem 22, with $K_1 = \{0\}$ and $K_2 = \mathbb{R}^{m_2}_+$, and maintaining the assumptions $g^1(x)$ linear affine, g^2 concave, $int(X) \neq \varnothing$, states that the following constraint qualifications, for the saddle points conditions of the Lagrangian function, are equivalent.

• $(CQ)_1$:
$i)$ $\exists\, x^* \in int(X)$, such that $g^1(x^*) = 0$;
$ii)$ $\exists\, x^{**} \in X$, such that $g^1(x^{**}) = 0$, $g^2(x^{**}) > 0$.

• $(CQ)_2$:
$\exists\, \bar{x} \in int(X)$, such that $g^1(\bar{x}) = 0$, $g^2(\bar{x}) > 0$.

• $(CQ)_3$:
$i)$ $\exists\, \hat{x} \in int(X)$, such that $g^1(\hat{x}) = 0$, $g^2(\hat{x}) \geqq 0$;
$ii)$ $\exists\, x^{**} \in X$ such that $g^1(x^{**}) = 0$, $g^2(x^{**}) > 0$.

Appendix 2 of Moore's paper (1968) corrects the error (or misprint) of Uzawa's paper. The counterexample of Moore is the following one:

$$\max f(x)$$

$$\text{subject to} \quad : \quad \begin{cases} x_1 + x_2 - 1 = 0 \\ -(2x_1 + x_2 - \frac{3}{2})^2 \geqq 0 \\ x_1 \geqq 0,\ x_2 \geqq 0. \end{cases}$$

This problem verifies the assumptions of Theorem 3 of Uzawa (1958), but there is no saddle point at the solution $x^0 = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Appendix 3 of Moore's paper treats the existence of a (Kuhn-Tucker) saddle point for (P_1) when the constraints are linear affine. The author points out the following remarks.

1) If also the objective function is linear affine (i. e. we have a Linear Programming Problem), then no constraint qualification is needed. This is a basic result in Linear Programming: see, e. g., Karlin (1959), Theorem 5.3.1.

2) If $f : D \longrightarrow \mathbb{R}$, D proper subset of $\mathbb{R}^n$, $h : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, h affine, f concave on X, $X \subset D$, X convex, the Lagrangian function $\mathfrak{L}(x, \lambda) = f(x) + \lambda^\top h(x)$ may fail to have a saddle point at the solution x^0 of the problem

$$\max f(x), \text{ subject to } h(x) \geqq 0,\ x \in X,$$

if no constraint qualification is imposed. Also for this statement Moore gives a numerical example. In Theorem 7 of the same paper Moore gives conditions to overcome the previous failure of the present point.

Subsequently, Kuhn-Tucker saddle points conditions for programming problems with linear affine constraints and, in some investigations, also with a set constraint, have been treated by various authors. See, e. g., Bazaraa (1973), Bazaraa and Shetty (1976), Bazaraa, Sherali and Shetty (1993), Berge and Ghouila-Houri (1962), Güler (2010), Jaffray and Pomerol (1989), Ponstein and Klein Haneveld (1975), Rockafellar (1970). We give here a quite simple proof of a generalization of the theorem of the alternative of Fan-Glicksberg-Hoffman (Theorem 3 of the present paper).

Theorem 23. Consider the following inequality system

$$\left\{ \begin{array}{c} f(x) < 0 \\ g_j(x) \leqq 0, \quad j = 1, ..., m \\ x \in C \end{array} \right. \tag{9}$$

where $f : C \longrightarrow \mathbb{R}$, $g_j : C \longrightarrow \mathbb{R}$, $j = 1, ..., m$, are convex functions on the convex set $C \subset \mathbb{R}^n$. Let us suppose that the following "generalized Slater condition" holds: there exists $x^0 \in relint(C)$ such that

$$\left\{ \begin{array}{c} g_j(x^0) < 0 \text{ for all } j \text{ such that } g_j \text{ is not linear} \\ g_j(x^0) \leqq 0 \text{ for all } j \text{ such that } g_j \text{ is linear.} \end{array} \right. \tag{10}$$

Then system (9) admits no solution if and only if there exists a vector $y^\top = [y_1, y_2, ..., y_m] \geqq 0$ such that

$$f(x) + \sum_{j=1}^{m} y_j g_j(x) \geqq 0, \quad \forall x \in C. \tag{11}$$

Before proving Theorem 23 we introduce some more symbols and present preliminary considerations. Let us define by F the following set

$$F \equiv \{x \in C : g_j(x) \leqq 0,\ j = 1, ..., m\}.$$

Some functions g_j may be identically zero on F; we call these functions "singular functions", while the others are called "regular functions". We introduce the following index sets.

$$\begin{aligned} J &\equiv \{1, 2, ..., m\}; \\ J_s &\equiv \{j \in J : g_j(x) = 0,\ \forall x \in F\}; \\ J_r &\equiv J \diagdown J_s = \{j \in J : g_j(x) < 0 \text{ for some } x \in F\}. \end{aligned}$$

We note that if the generalized Slater condition holds, then all singular functions g_j must be linear. We recall the following classical separation theorem (see, e. g., Mangasarian (1969), Rockafellar (1970)) which does not require that the set $U \subset \mathbb{R}^n$ (involved in the theorem) is closed.

Lemma 1. Let $U \subset \mathbb{R}^n$ be a convex set and a point $w \in \mathbb{R}^n$ with $w \notin U$ be given. Then there exists a separating hyperplane $\{x : a^\top x = \alpha\}$, with $a \in \mathbb{R}^n$, $a \neq 0$, $\alpha \in \mathbb{R}$, such that

1. $a^\top w \leqq \alpha$;
2. $a^\top u \geqq \alpha$ for all $u \in U$, but U is not a subset of the hyperplane.

Note that the last property says that there is a vector $\bar{u} \in U$ such that $a^\top \bar{u} > \alpha$.

Proof of Theorem 23. If (9) admits a solution, obviously (11) cannot hold for that solution. This is the trivial part of the proof, which holds without convexity (or generalized convexity) assumption and without the generalized Slater condition. Now, let us assume that (9) admits no solution. With $u^\top = [u_0, u_1, ..., u_m] \in \mathbb{R}^{m+1}$, let us define the following set

$$U \equiv \{u : \exists x \in C \text{ such that } u_0 > f(x),\ u_j \geqq g_j(x) \text{ if } j \in J_r,\ u_j = g_j(x),\ \text{if } j \in J_s\}.$$

Clearly, U is nonempty and convex. Because of our supposition, U does not contain the origin of $\mathbb{R}^{m+1}$. Therefore, thanks to Lemma 1, there exists a separating hyperplane defined by the nonzero vector $[y_0, y_1, ..., y_m]$ such that

$$\sum_{j=0}^{m} y_j u_j \geqq 0,\ \forall u \in U \tag{12}$$

and

$$\sum_{j=0}^{m} y_j \bar{u}_j > 0,\ \text{for some } \bar{u} \in U. \tag{13}$$

We perform the remaining proof in three steps:

(I) First we prove that $y_0 \geqq 0$ and $y_j \geqq 0$ for all $j \in J_r$.

(II) Secondly we establish that (12) holds for $u = (f(x), g_1(x), ..., g_m(x))^\top$, if $x \in C$.

(III) Thirdly we prove that $y_0 > 0$.

Proof of (I). We show that $y_j \geqq 0$ for all $j \in \{0\} \cup J_r$. Let us assume that $y_0 < 0$. Let us take an arbitrary vector $(u_0, u_1, ..., u_m)^\top \in U$. By definition $(u_0 + \lambda, u_1, ..., u_m)^\top \in U$ for all $\lambda \geqq 0$. Hence by (12) one has

$$\lambda y_0 + \sum_{j=1}^{m} y_j u_j \geqq 0 \text{ for all } \lambda \geqq 0.$$

For sufficiently large λ the left hand side of the last inequality is negative, which is a contradiction. Therefore it holds $y_0 \geqq 0$. The proof of the nonnegativity of all other y_j, $j \in J_r$, is similar.

Proof of (II). Secondly, we establish that

$$y_0 f(x) + \sum_{j=1}^{m} y_j g_j(x) \geqq 0,\ \forall x \in C. \tag{14}$$

This follows from the remark that for all $x \in C$ and for all $\lambda \geqq 0$ one has $u = (f(x) + \lambda, g_1(x), ..., g_m(x))^\top \in U$, thus

$$y_0(f(x) + \lambda) + \sum_{j=1}^{m} y_j g_j(x) \geqq 0,\ \forall x \in C.$$

Taking the limit as $\lambda \longrightarrow 0$ the claim follows.

Proof of (III). Thirdly we show that $y_0 > 0$. The proof is by contradiction. We already know that $y_0 \geqq 0$. Let us assume $y_0 = 0$. Hence, from (14) we have

$$\sum_{j \in J_r} y_j g_j(x) + \sum_{j \in J_s} y_j g_j(x) = \sum_{j=1}^{m} y_j g_j(x) \geqq 0, \text{ for all } x \in C.$$

Taking a point $x^* \in relint(C)$ such that

$$\begin{array}{rcl} g_j(x^*) & < & 0, \ \forall j \in J_r \\ g_j(x^*) & = & 0, \ \forall j \in J_s \end{array}$$

(this point surely exists thanks to (10)), one has

$$\sum_{j \in J_r} y_j g_j(x^*) \geqq 0.$$

Since $y_j \geqq 0$ and $g_j(x^*) < 0$ for all $j \in J_r$, this implies $y_j = 0$ for all $j \in J_r$. Therefore it holds

$$\sum_{j \in J_s} y_j g_j(x) \geqq 0, \ \forall x \in C. \tag{15}$$

Now, from (13), with $\bar{x} \in C$ such that $\bar{u}_j = g_j(\bar{x})$ for $j \in J_s$, we have

$$\sum_{j \in J_s} y_j g_j(\bar{x}) > 0. \tag{16}$$

Because $x^* \in relint(C)$, there exist a vector $\bar{\bar{x}} \in C$ and $\lambda \in (0,1)$ such that $x^* = \lambda \bar{\bar{x}} + (1-\lambda)\bar{x}$. Taking into account that it holds $g_j(x^*) = 0$ for all $j \in J_s$ and that the singular functions are linear, one gets

$$\begin{array}{rcl} 0 & = & \displaystyle\sum_{j \in J_r} y_j g_j(x^*) = \sum_{j \in J_s} y_j g_j(\lambda \bar{\bar{x}} + (1-\lambda)\bar{x}) = \\ & = & \displaystyle\lambda \sum_{j \in J_s} y_j g_j(\bar{\bar{x}}) + (1-\lambda) \sum_{j \in J_s} y_j g_j(\bar{x}) > (1-\lambda) \sum_{j \in J_s} y_j g_j(\bar{\bar{x}}). \end{array}$$

The last inequality follows from (16). Since $(1-\lambda) > 0$ we obtain the inequality

$$\sum_{j \in J_s} y_j g_j(\bar{\bar{x}}) < 0$$

which contradicts (15). Hence we have proved that $y_0 > 0$.

At this point we have (14), with $y_0 > 0$ and $y_j \geqq 0$ for all $j \in J_r$. Dividing by $y_0 > 0$ in (14) and by redefining $y_j \equiv (y_j / y_0)$ for all $j \in J$ we have the desired result. □

Remark 1. Of course the multipliers of all singular constraints can always be chosen strictly positive.

Remark 2. Theorem 23 enables us to obtain the classical *Farkas-Minkowski theorem,* whose proof usually requires to show that a *finite cone* is a closed (convex) set. This fact is given for granted in several books, but it must be proved and the related proof is not difficult, but not trivial. The classical Farkas-Minkowski theorem of the alternative is stated as follows.

• For any given matrix A of order (m, n) and any given vector $b \in \mathbb{R}^m$ one and only one of the following two linear systems admits a solution

$$\begin{aligned} S_1 \quad &: \quad Ax = b, \ x \geqq 0 \\ S_2 \quad &: \quad A^\top u \geqq 0, \ b^\top u < 0. \end{aligned}$$

This result can be simply deduced from Theorem 23. It is quite immediate to verify that S_1 and S_2 cannot have both a solution. It remains to prove that if S_2 has no solutions, then S_1 admits a solution. Let us write S_2 in the form $f(u) \equiv b^\top u < 0$, $-A^\top u \leqq 0$. If S_2 admits no solution, from Theorem 23 we have that there exists $x \in \mathbb{R}^m_+$ such that

$$b^\top u - x^\top A^\top u = u^\top (b - Ax) \geqq 0,$$

for all $u \in \mathbb{R}^m$, and therefore it holds $b - Ax = 0$, $x \geqq 0$. □

Remark 3. There are in the literature many versions of nonlinear theorems of the alternative, with various degrees of generality. See. e. g., Bazaraa (1973), Cambini (1986), Giannessi (1984), Jeyakumar (1985), Kolumban (1997), Oettli and Gwinner (1994). In particular, the following result is a special case of a more general theorem proved by Cambini (1986) and Giannessi (1980, 1984).

Theorem 24. Let $C \subset \mathbb{R}^n$ be a nonempty convex set and let $f : C \longrightarrow \mathbb{R}^p$, $g : C \longrightarrow \mathbb{R}^q$ be convex functions. Then:

(i) If the system

$$\begin{cases} f_k(x) < 0, \ k = 1, ..., p, \\ g_j(x) \leqq 0, \ j = 1, ..., q, \end{cases} \tag{17}$$

admits no solutions, then there exist vectors $u^\top \in \mathbb{R}^p_+$, $v^\top_+ \in \mathbb{R}^q_+$, with $(u, v)^\top \neq 0$, such that

$$u^\top f(x) + v^\top g(x) \geqq 0, \ \forall x \in C. \tag{18}$$

(ii) If (18) holds with $u^\top \in \mathbb{R}^p_+$, $v^\top_+ \in \mathbb{R}^q_+$, $(u, v)^\top \neq 0$ and, moreover, it holds

$$\left\{x \in C, \ f_k(x) < 0, \ k = 1, ..., p; \ g_j(x) \leqq 0, \ j = 1, ..., q, \ v^\top g(x) = 0\right\} = \varnothing,$$

whenever $u = 0$, then system (17) is impossible.

From this theorem Giannessi (1980) obtains in a simple way a nonhomogeneous version of the Farkas-Minkowski theorem, due to Duffin (see Mangasarian (1969)). From the Duffin theorem we get at once the Farkas-Minkowski theorem.

From Theorem 23 it is easy to obtain the Kuhn-Tucker saddle points conditions for the problem

$$(P_2): \qquad \begin{cases} \min f(x) \\ \text{subject to: } g_i(x) \leqq 0, \ i = 1, ..., m, \\ h_j(x) \leqq 0, \ j = 1, ..., p, \\ x \in X, \end{cases}$$

where $X \subset \mathbb{R}^n$ is convex, $f : X \longrightarrow \mathbb{R}$, $g_i : X \longrightarrow \mathbb{R}$, $i = 1, ..., m$, are convex and h_j is linear affine for all $j = 1, ..., p$. Here the Lagrangian function is

$$\mathfrak{L}(x, u, w) = f(x) + u^\top g(x) + w^\top h(x),$$

with $u \geqq 0$, $w \geqq 0$; the related generalized Slater condition is

$$(S_1) \quad \exists \bar{x} \in relint(X) : g(\bar{x}) < 0, h(\bar{x}) \leqq 0.$$

Taking into account that the equality system $h(x) = 0$ is equivalent to $h(x) \leqq 0$, $-h(x) \leqq 0$, always from Theorem 23 it is easy to obtain saddle points conditions for the problem

$$(P_3): \qquad \begin{cases} \min f(x) \\ \text{subject to: } g_i(x) \leqq 0, \ i = 1, ..., m, \\ h_j(x) = 0, \ j = 1, ..., p, \\ x \in X, \end{cases}$$

where X, f, g and h verify the same assumptions of before. Here the Lagrangian function is

$$\mathfrak{L}(x, u, w) = f(x) + u^\top g(x) + w^\top h(x),$$

with $u \geqq 0$, $w \in \mathbb{R}^p$, and the generalized Slater condition is

$$(S_2) \quad \exists \bar{x} \in relint(X) : g(\bar{x}) < 0, h(\bar{x}) = 0.$$

A further weakening of (S_1) or (S_2) consists in considering only the *active constraints* at the solution x^0 of the related problem. For example, with reference to (P_2) :

$$(S_3) \quad \exists \bar{x} \in relint(X) : g_i(\bar{x}) < 0, \ \forall i \in I(x^0), \ h(\bar{x}) \leqq 0.$$

Furthermore, also for (P_2) and (P_3) it is easy to obtain conditions equivalent to the related saddle points conditions, as previously done for (P): see Theorem 5. For example, with reference to (P_2) we have the following conditions, where $x^0 \in X$, $u^0 \geqq 0$, $w^0 \geqq 0$.

(α) $\mathfrak{L}(x^0, u^0, w^0) \leqq \mathfrak{L}(x, u^0, w^0)$, $\forall x \in X$.

(β) $g(x^0) \leqq 0$, $h(x^0) \leqq 0$.

(γ) $(u^0)^\top g(x^0) = 0$; $(w^0)^\top h(x^0) = 0$.

We note that the Slater constraint qualification for (P_3) considered by Moore (1968) and previously briefly discussed, is slightly less general than (S_2), as it has the form

$$(S_4) \quad \exists \bar{x} \in int(X) : g(\bar{x}) < 0, \ h(\bar{x}) = 0,$$

which obviously requires that $int(X) \neq \varnothing$.

It is also possible, as done, e. g. by Ruszcsynski (2006), to obtain for (P_3), or also for (P_2), optimality conditions expressed in terms of subdifferentials. For the reader's convenience we report the following result (Theorem 3.34 of Ruszcsynski (2006)).

Theorem 25. Assume that the point x^0 is a solution of problem (P_3), $f(x)$ and $g_i(x)$, $i = 1, ..., m$, are convex on an open convex set containing the feasible set, the function $h(x)$ is linear affine and the generalized Slater condition (S_3) is satisfied. Then there exist $u^0 \in \mathbb{R}^m_+$ and $w^0 \in \mathbb{R}^p$ such that

$$0 \in \partial f(x^0) + \sum_{i=1}^{m} u_i^0 \partial g_i(x^0) + \sum_{j=1}^{p} w_j^0 \nabla h_j(x^0) + N(X, x^0);$$

$$u_i^0 g_i(x^0) = 0, \quad i = 1, ..., m,$$

$N(X, x^0)$ being the *normal cone of* X *at* x^0. Conversely, if for some feasible point of (P_3) the above conditions are satisfied, then x^0 is a solution of problem (P_3).

Theorem 25 is a generalization of Theorem 13 of the present paper. Finally, we note that Bazaraa (1973), Bazaraa and Shetty (1976) and Bazaraa, Sherali and Shetty (1993) take into consideration saddle points optimality conditions for problem (P_3) and impose the following constraint qualification, due to Neustadt (1970).

$$(N) \quad \exists \bar{x} \in X : g(\bar{x}) < 0, \ h(\bar{x}) = 0 \text{ and } 0 \in int(h(X)),$$

where $h(X) = \{h(x) : x \in X\}$.

Under the assumption $int(X) \neq \varnothing$, $int(h(X)) \neq \varnothing$, it can be proved that (N) and (S_4) are equivalent conditions. In absence of the above assumption the two constraint qualifications are independent conditions (Jiménez and Novo (2024)).

- Let us show that $(N) \nRightarrow (S_4)$. Take into consideration in $\mathbb{R}^2$, $X = [-1, 1] \times \{0\}$, $g(x_1, x_2) = x_1^2 + x_2^2 - 1$ and $h(x_1, x_2) = x_1$. Then (N) is satisfied, because $h(X) = [-1, 1]$ and thus $0 \in int(h(X))$. Moreover, choosing $x^* = (0, 0)$ one has $x^* \in X$, $g(x^*) < 0$ and $h(x^*) = 0$. But (S_4) is not satisfied, as $int(X) = \varnothing$.
- Let us show that $(S_4) \nRightarrow (N)$. Take into consideration in $\mathbb{R}^2$, $X = [-1, 1] \times [-1, 1]$, $g(x_1, x_2) = (x_1^2 + x_2^2 - 1)$ and $h(x_1, x_2) = (x_1, x_2, 0)$. Then (S_4) is satisfied, because by choosing $x^* = (0, 0) \in int(X)$ we have $g(x^*) < 0$ and $h(x^*) = 0$. But (N) is not satisfied because $0 \notin int(h(X))$, since $h(X) = [-1, 1] \times [-1, 1] \times \{0\}$ and so $int(h(X)) = \varnothing$.

Therefore (S_4) and (N) are two independent constraint qualifications. Now, a natural question arises: what happens if $int(X) \neq \varnothing$, since in (S_4) this condition is satisfied. Also, it is natural to require $int(h(X)) \neq \varnothing$, since in (N) this condition is satisfied. We have the following result.

Theorem 26.
(i) If $int(X) \neq \varnothing$, then $(N) \Longrightarrow (S_4)$.
(ii) If $int(h(X)) \neq \varnothing$, then $(S_4) \Longrightarrow (N)$.
(iii) If $int(X) \neq \varnothing$ and $int(h(X)) \neq \varnothing$, then (S_4) and (N) are equivalent.

Proof.
(i) Assume that (N) holds. By Theorem 6.6 in Rockafellar (1970) one has $h(relint(X)) = relint(h(X))$. As $int(X) \neq \varnothing$ by hypothesis, and $int(h(X)) \neq \varnothing$, since (N) holds, it follows that $h(int(X)) = int(h(X))$. By assumption $0 \in int(h(X)) = h(int(X)))$, therefore there exists $x^1 \in int(X)$ such that $h(x^1) = 0$. Now we make a classic reasoning to find a point satisfying (S_4). Let $x_t = tx^1 + (1-t)x^*$, with $0 \leqq t \leqq 1$, x^* being a point that satisfies (N). Then firstly, $h(x_t) = 0$ for all t; secondly, $x_t \in int(X)$, since X is convex and $x^1 \in int(X)$, $x^* \in X$, and thirdly, as g is convex we have $g(x_t) \leqq tg(x^1) + (1-t)g(x^*) = g(x^*) + t(g(x^1) - g(x^*)) \longrightarrow g(x^*) < 0$ when $t \longrightarrow 0$. So, for t small enough one has $g(x_t) < 0$, and therefore these points x_t satisfy condition (S_4).

(ii) Assume that (S_4) holds. This part is obvious since under our assumptions $h(int(X)) = int(h(X))$ and so for a point x^* satisfying (S_4) we have $0 = h(x^*) \in int(h(X))$, since $x^* \in int(X)$, and consequently (N) holds.

(iii) It is a consequence of points (i) and (ii). □

In the previous theorem the condition $int(h(X)) \neq \varnothing$ is not too natural. Can we provide an equivalent or at least a sufficient condition? We can use Lemma 3.2 in Jiménez and Novo (2002), but we need a convex set X' with $0 \in X'$. As $int(h(X)) \neq \varnothing$, take $b \in int(h(X))$, $q^0 \in X$ such that $h(q^0) = b$ and let $X' = X - q^0$. This set satisfies:

1) $0 \in X'$;

2) $0 \in int(h(X'))$. Indeed $h(X') = h(X - q^0) = h(X) - b$ and $b \in int(h(X))$.

Now we can apply Lemma 3.2 of Jiménez and Novo (2002), to obtain several equivalent conditions to have $0 \in int(h(X'))$. We let the interested reader to develop these conditions and express them in terms of X instead of X'. We point out that the equivalence between (S_4) and (N) is stated in Giorgi (2002) without the precisations of Theorem 26.

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